*Original Research Article*

# Research on the numerical solution and dynamic properties of nonlinear fractional differential equations

**Na Wang, Yanqin Yang**

Shanxi Technology and Business College, Shanxi, China, 030000

***Abstract:*** Fractional calculus is an important branch of mathematical analysis, which is specialized in the study of the mathematical properties and applications of arbitrary order integral and differential, and is the extension of the traditional integral calculus. At present, fractional integral and derivative operators are mainly used to calculate fractional calculus, among which the most famous ones are Riemann-Liouville fractional integral and derivative, Caputo fractional derivative, Grümwald-Letnikov fractional integral and derivative, etc. At present, the numerical algorithm of finite difference scheme is mainly used to solve the approximate solution of the equation, to solve the fractional differential equation. Through the finite difference of time fractional order or space fractional order, the approximate solution of the equation is obtained, and the stability, convergence and compatibility of the scheme are checked, and the convergence order and estimation error are calculated. At present, the theory and method of nonlinear fractional differential equation are widely used in the study of various intermediate processes and critical phenomena in finance, physics and mechanics, which can better fit some natural physical processes and dynamic system processes.

***Keywords:*** Fractional calculus; numerical solution of nonlinear fractional differential equations; Grümwald-Letnikov, Riemann-Liouville, Caputo, Dynamics

## 1. Introduction

The theory of fractional calculus is an important branch of mathematical analysis. It is specialized in the study of the mathematical properties and applications of arbitrary order integrals and differential calculus. It is the extension of traditional integral order calculus.

Compared with the development of integral calculus, the development of fractional calculus also consumes a lot of research energy, the theory of fractional calculus has experienced a long and tortuous development process for many years. At present, fractional integral and derivative operators are mainly used to calculate fractional calculus,, among which the most famous ones are Riemann-Liouville fractional integral and derivative, Caputo fractional derivative, Grümwald-Letnikov fractional integral and derivative, etc. At present, the numerical algorithm of finite difference scheme is mainly used to solve the approximate solution of the equation, to solve the fractional differential equation. Through the finite difference of time fractional order or space fractional order, the approximate solution of the equation is obtained, and the stability, convergence and compatibility of the scheme are checked, and the convergence order and estimation error are calculated.

In recent years, many professional scholars for numerical solution of fractional differential equations for various research, makes the fractional order differential equation of related research is increasingly mature, fractional


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differential equation in biology, materials science, chemical kinetics, diffusion of electromagnetism, transmission, automatic control, etc. Also more and more widely applied in many fields of science.

# 2. Basic definition of fractional differential equation

Since the birth of fractional calculus in the 17th century, mathematicians have been exploring the theoretical system of fractional operators. After the efforts of many mathematicians, from different angles, we have established a variety of different forms of fractional order operator definitions. Now the main three kinds of definitions are as follows.

## 2.1 Definition of Grümwald-Letnikov fractional

For any real number α, the integral part [α] ([α] is the largest integer less than α.

If the function f (T) has m + 1-order continuous reciprocal on the interval [α, t], when α > 0, m is at least [α], then the fractional derivative α is defined as:

$$ {}_a^G D_t^\alpha f(t) \triangleq \lim_{h\to 0} h^{-a} \sum_{i=0}^{n} [{}^{-\alpha}_{i}] f\ (t-ih) $$

Among them, $[{}^{-\alpha}_{i}]=\frac{(-\alpha)\ (-\alpha+1)\ (-\alpha+2)\cdots\cdots(-\alpha+i-1)}{i!}$, so the above formula can be changed into:

$$ {}_a^G D_t^\alpha f(t) \triangleq \sum_{i=0}^{n} \frac{f^{(k)}(\alpha)\ (t-\alpha)^{-\alpha+k}}{\Gamma\ (-\alpha+k+1)} + \frac{1}{\Gamma\ (-\alpha+m+1)} \int_0^t (t-\tau)^{-\alpha+m} f^{(m+1)}\ (\tau)\ d\tau $$

This definition is derived from looking for the unity of n derivative and n integral. The n can be extended to a negative integer.

## 2.2 Definition of Riemann-Liouville fractional

The fractional derivative of function definition is:

$$ {}_a^R D_t^\alpha f(t) = \begin{cases} \frac{d^n f}{dt^n}, \alpha = n \in N \\ \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,\ 0 \leqslant n-1 < \alpha < n \end{cases} $$

Similarly, the fractional integral of Riemann-Liouville fractional definition is:

$$ {}_a^R D_t^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} f\ (\tau)\ d\tau, (\alpha < 0) $$

Under the condition that the function has m + 1 order continuous derivative and m is at least [α]=n-1, the definition of Grümwald-Letnikov fractional is equivalent to that of Riemann-Liouville fractional. But without the above conditions, the definition of Riemann-Liouville fractional is an extension of the definition of Grümwald-Letnikov fractional, and its application range is more extensive.

## 2.3 Definition of Caputo fractional

For positive non integer α, (the rest is the same as the definition of Riemann-Liouville fractional),

$$ {}_a^c D_t^\alpha f(t) \triangleq \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n\ (\tau)\ d\tau $$
$$ 0 \le n-1 < \alpha < n,\ n \in N $$

Both the definition of Caputo fractional and the Riemann-Liouville fractional are improvements to the definition of Grümwald-Letnikov fractional.

Under the condition:

(1) if $f(t)$ has m + 1 order continuous derivative, and m is at least [α]=n-1;

(2) $f^{(k)}(a)=0, k=0,1,2,\cdots n-1,$ the definition of Caputo fractional is equivalent to that of Riemann-Liouville fractional.

# 3. Numerical solution of nonlinear fractional differential equation

In recent years, the numerical solutions of nonlinear fractional differential equations can be divided into two directions, one is for fractional ordinary differential equations, the other is for fractional partial differential equations. In time or space, the derivative of the original integer order is replaced by fractional order, and the fractional order differential equation of time or space is obtained, which is used to describe the physical and mechanical motion that the integral order differential equation cannot describe.

At present, the solution of fractional order differential equation mainly uses the numerical algorithm of finite difference scheme to solve the approximate solution of the equation. Through the finite difference of time fractional order or space fractional order, the approximate solution of the equation is obtained, and the stability, convergence and compatibility of the scheme are checked, and the convergence order and estimation error are calculated.

For one-dimensional space fractional differential equation, Vincent J. Ervin *et al.* Discussed a finite element numerical approximation method for solving time-dependent quadratic nonlinear fractional diffusion equation. First, the theoretical hypothesis was given, then the correctness was proved by numerical examples, and the convergence and prior error estimates of the approximate values were given. Bois baeume *et al.* Developed a practical method for solving fractional order reaction-diffusion equations. The method is based on operator splitting, and the results are shown by images, and its application in biology is discussed.

Tadjeran *et al.* proposed an accurate and effective numerical method for the solution of two-dimensional spatial fractional differential equations, combining alternating direction implicit approximation, C-N discretization and Richardson extrapolation, and finally obtained a finite difference method with stability and compatibility, and achieved a relatively stable degree of second-order accuracy.

Space of fractional order ordinary differential equation solution, Kai Diethelm and other professional scholars have adopted an Adams model forecast - correction methods to solve the fractional order differential equation, analyses the existence of nonlinear fractional differential equations, uniqueness and stability of structure, and by adopting Caputo differential form, a detailed estimate of error, the error of gradual expansion has carried on the simple instructions Liu Fawang *et al.* Considered the simplest fractional order ordinary differential equation, introduced the fractional order linear multistep method, derived the higher order approximation of the initial value problem of fractional order ordinary differential equation, proved the consistency and convergence of the method, and gave the stability analysis, considered the fractional order relaxation operation equation, proved the existence and uniqueness of the solution of the equation, and gave the solution by using Green's function Its analytical solution is given, and an effective fractional order predictor corrector method is proposed.

# 4. Dynamics of nonlinear fractional differential equations

In recent years, with the development of computer, the improvement of computer technology and its application, as well as the vigorous development of modern mathematical technology, the rich steady motion and complex mixed motion in nonlinear system are constantly revealed, which attracts more and more theoretical and application workers' extensive attention. However, the spatial structure of the system hybrid attractor and its related graphics is extremely complex, rich in connotation and many changes in form, which has also attracted extensive attention and more and more in-depth research in various fields. It has become a research hotspot not only in theory, but also in application. With the continuous development of scientific visualization technology, chaos attractor and its related graphics have begun to attract attention in the fields of advertising, printing, arts and crafts. Textile pattern design belongs to the traditional field of Arts and crafts, also belongs to the category of visual arts.

In 2006, Chen *et al.* Studied parallel bar chaotic attractors based on integer order Rossler dynamical systems. Then they show that the same phenomenon exists in other continuous power systems. In 2009, Wu Jianxin *et al.* Obtained the general method of generating parallel strip chaotic attractors and rectangular chaotic attractors for general integer order nonlinear dynamical systems.

## 5. Conclusions

The development of fractional calculus has a history of more than 300 years. It is actually an extension of the concept of integral calculus. In the early stage, on the one hand, it is different from the classical integer system in physics, on the other hand, it has no application background in physics and mechanics, so the development is very slow. Until the middle and late of the century, fractional calculus, as a basic concept of fractal geometry and fractal dimension, developed rapidly. Because of its nonlocality, fractional calculus operators were used to describe the transport dynamics of materials with memory and genetic properties and complex systems controlled by anomalous diffusion in the real world. In recent decades, fractional calculus and fractional differential equations have been applied in many fields, such as oil seepage, groundwater pollution control, viscoelastic materials, signal and image processing, control, quantum mechanics, viscoelastic dynamics, random walk, finance, life science and so on. Because most of the analytical solutions of fractional order differential equations are difficult to be obtained by analytical methods, the study of approximate solutions of fractional order differential equations becomes particularly important.

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