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Higher Order Willmore Revolution Hypersurfaes in R^{n+1} *

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Abstract: Let $x: M^n \to N^{n+1}$ be an n-dimensional hypersurface immersed in an $(n+1)$-dimensional Riemannian manifold N^{n+1}. Let σ_i $(0 \leq i \leq n)$ be the ith mean curvature and $Qn = \sum_{r=0}^{n} (-1)^{i+1} C_n^i \sigma_1^{n-i} \sigma_i$, where C_n^i is binomial coefficient. The second author showed that functional $Wn(x)=\int_M Q_n dM$ is a conformal invariant and gave the Euler-Lagrange equation. W_n is called the nth Willmore functional of x. A hypersurface is called the nth order Willmore hypersurfaces if it is a solution of the Euler-Lagrange equation. In this paper, we establish the ordinary differential equation of the nth order Willmore revolution hypersurfaces and present standard examples of the nth order Willmore hypersurfaces.

Key words: Conformal invariants; *n*th order Willmore functional; *n*th order Willmore revolution hypersurface.

1. Introduction

Let $x: M^n \to N^{n+1}$ be an n -dimensional hypersurface isometrically immersed in a Riemannian (n+1)-manifold N^{n+1}. Let g_{ij} be the Riemannian metric tensor of M^n at x and h_{ij} be the second fundamental tensor of M^n at x. The roots of the equation $det(h_{ij} - \lambda g_{ij}) = 0$, $\lambda_1, \cdots, \lambda_n$ are called the principal curvature of M^n at x, and the rth mean curvature σ_r of M^n at x is defined dy

$$\sigma_r = \frac{1}{C_n^r} \left(\sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \right), \quad r = 1, \cdots, n,$$

where C_n^r is the binomial coefficient. For convenience, we define σ_0=1. When n=2 and N is 3-dimensional Euclidean space R^3, the functional is reduced to $W(M) = \int_M \sigma_1^2 \, dM$. The variational problem of the functional was studied by G. Thomsen and W. Blaschke around 1923[2]. In 1965, T. J. Willmore restudied the functional, and proved that a closed pipe-like surface which is generated by a circle revolving around a closed curve satisfies $W(M) > 2\pi^2$ and the equality holds if and only if M is a standard torus. Then he proposed the famous Willmore Conjecture: the above inequality holds for all topological tori [16] and the equality holds if and only if the surface conformally equivalent to the standard torus. Recently, F. Margues and A. Neves resolved this conjecture . There are two aspects of generalization for classical Willmore functional. One is for general dimension and codimension of M. In 1973, B. Y. Chen proved that the functional.

$$W(M) = \int_M (\sigma_1^2 - \sigma_2)^{\frac{n}{2}} dM, \tag{1.1}$$

is a conformal invariant[5]. Z. Guo, H. Li and C. Wang gave the formulas of the first and the second variational of W defined by (1.1). They also present some standard examples of the stable critical hypersurfaces of the functional[7]. Another aspect of the generalization is for the orders explanted in the following. Let M^n be a hypersurface of N^{n+1}, for any integer r , $2 \leq r \leq n$. It was proved that the functional

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$$W_r(M)=\begin{cases}\int_M Q_r^{\frac{n}{r}}dM, & r<n, \text{and } r \text{ is odd},\\ \int_M |Q_r|^{\frac{n}{r}}dM, & r<n, \text{and } r \text{ is even},\\ \int_M Q_r dM, & r=n,\end{cases} \tag{1.2}$$

is a conformally invariant of M^n in N^{n+1} , where Qr $=\sum_{k=0}^{r}(-1)^{k+1}C_r^k\sigma_1^{r-k}\sigma_k$. As for $r = 2$ W_r recovers the the Chen-Willmore functional, we call $W_r(M)$ rth order Willmore functional[8].

Define the kth Newston transformation $T_{(k)}$ of $A = (h_{ij})n\times n$ by:

$$T_{(k)}= S_kI - S_{k-1}A +\cdots+ (-1)^{k-1} S_1A^{k-1} + (-1)^k A^k , k = 0, 1,\cdots, n.$$

Where $S_k = C_n^k\sigma_k$. The second author calculated the variation of functional $W_r(M)$, and got the Euler-Lagrange equation as follows[8] :

$$\begin{aligned}&\Delta(Q_r^{\frac{n-r}{r}}(Q_{r-1}+\sigma_1^{r-1}))+(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}T_{(k-1)ij}(Q_r^{\frac{n-r}{r}}\sigma_1^{r-k})_{,ij}\\&+Q_r^{\frac{n-r}{r}}(n^2\sigma_1^2-n(n-1)\sigma_2+nc)(Q_{r-1}+\sigma_1^{r-1})\\&-n\sigma_1Q_r^{\frac{n}{r}}+Q_r^{\frac{n-r}{r}}(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k\\&-(n-k)\sigma_{k+1}+ck\sigma_{k-1})=0.\end{aligned} \tag{1.3}$$

In particular, when r = n, we have:

$$\begin{aligned}&\Delta(Q_{n-1}+\sigma_1^{n-1})+\sum_{k=2}^{n}(-1)^{k+1}T_{(k-1)ij}(\sigma_1^{n-k})_{,ij}+(n^2\sigma_1^2-n(n-1)\sigma_2\\&+nc)(Q_{n-1}+\sigma_1^{n-1})-n\sigma_1Q_n+\sum_{k=2}^{n}(-1)^{k+1}C_n^k\sigma_1^{n-k}(n\sigma_1\sigma_k\\&-(n-k)\sigma_{k+1}+ck\sigma_{k-1})=0.\end{aligned} \tag{1.4}$$

Where c is the sectional curvature of N . Here we call a hypersurface satisfying (1.5) the nth order Willmore hypersurface. When $n = r = 3$, the second author gave the standard examples of 3th order Willmore hypersurfaces.

The main purpose of this paper is to construct the nth order Willmore hypersurface for general dimension $n \geq 3$. We organize the paper as follows: in Section 2 we establish the ordinary differential equation of nth order Willmore revolution hypersurfae in R^{n+1} (see Theorem 2.1); in Section 3 we consider the solution of the equation and get the standard examples of nth order Willmore hypersurfaces (see Theorem 2.2).

2. O. D. E of *n*th order Willmore revolution hypersurfae in R^{n+1}

2.1 Some lemmas

Let S^{n-1} be the unit sphere in n-dimensional Euclidean space R^n , $\xi = (\xi1 ,\cdots, \xi_n)$ be the position vector of a point of S^{n-1} in R^n and $\gamma = (\alpha, \beta) : R^1 \to R^2$ be a smooth curve in R^2 . We consider product immersion $\varphi : R^1 \times S^{n-1} \to R^{n+1}$, i. e. the revolution hypersurface which is defined by

$$\varphi(t, \xi) = (\alpha\xi, \beta). \tag{2.1.1}$$

We take local field of orthonormal tangent frames $\{\tilde{e}_a, 2 \leq a \leq n\}$ on S^{n-1} , its field of dual frames $\{\tilde{w}_a\}$ and make the following convention on the range of indices:

$$2 \leq a, b, c,\cdots, \leq n;\ \ 1 \leq i, j, k,\cdots, \leq n.$$

Then we can write the structure equations of S^{n-1} as follows:

$$d\xi=\sum_a\tilde{\omega}_a\tilde{e}_a, \tag{2.1.2}$$

$$d\tilde{e}_a = \sum_b \tilde{\omega}_{ab}\tilde{e}_b - \tilde{\omega}_a \xi. \tag{2.1.3}$$

Obviously, $\{\frac{d}{dt}, \tilde{e}_2, \cdots, \tilde{e}_n\}$ is a local field of tangent frames on $R^1 \times S^{n-1}$. From

$$\phi_*(\frac{d}{dt}) = \frac{d}{dt}(\phi) = (\alpha'\xi, \beta'),$$

$$\phi_*(\tilde{e}_a) = \tilde{e}_a(\phi) = (\alpha\tilde{e}_a, 0)$$

we see that if we take

$$e_1 = \frac{1}{|\gamma'|}(\alpha'\xi, \beta'), \quad e_a = (\tilde{e}_a, 0), \quad e_{n+1} = \frac{1}{|\gamma'|}(-\beta'\xi, \alpha') \tag{2.1.4}$$

then $\{e_1, \cdots, e_n\}$ forms a local field of orthonormal tangent frames on $R^1 \times S^{n-1}$, and e_{n+1} is a normal frame on hypersurface φ .

Let $\{\omega_1, \cdots, \omega_n\}$ be the dual frame of $\{e_1, \cdots, e_n\}$, then the structure equations of φ can be written as:

$$d\phi = \omega_1 e_1 + \sum_a \omega_a e_a,$$

$$de_1 = \sum_a \omega_{1a} e_a + \sum_j h_{1j}\omega_j e_{n+1},$$

$$de_a = \omega_{a1} e_1 + \sum_b \omega_{ab} e_b + \sum_j h_{aj}\omega_j e_{n+1},$$

$$de_{n+1} = -\sum_j h_{1j}\omega_j e_1 - \sum_{a,j} h_{aj}\omega_j e_a.$$

Where ω_{ij}, h_{ij} are called the connection form and the second fundamental tensor of M^n , which satisfy $\omega_{ij}+\omega_{ji} = 0$ and $h_{ij} = h_{ji}$. Taking exterior derivative the above equations and contrasting with (2.1.2) and (2.1.3), we have

$$\omega_1 = |\gamma'|dt, \quad \omega_a = \alpha\tilde{\omega}_a, \tag{2.1.5}$$

$$\omega_{1a} = \frac{\alpha'}{\alpha|\gamma'|}\omega_a, \quad \omega_{ab} = \tilde{\omega}_{ab},$$

$$h_{11} = \frac{\alpha'\beta'' - \alpha''\beta'}{|\gamma'|^3}, \quad h_{1a} = 0, \ h_{ab} = \frac{\beta'}{\alpha|\gamma'|}\delta_{ab}.$$

For convenience , we denote

$$\lambda = \frac{\alpha'\beta'' - \alpha''\beta'}{|\gamma'|^3}, \quad \mu = \frac{\beta'}{\alpha|\gamma'|}. \tag{2.1.6}$$

For a smooth function f on $R^1 \times S^{n-1}$, we define the first, second covariant derivatives and Laplacian as follows:

$$df = \sum_i f_i\omega_i, \ f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}, \ \Delta f = \sum_i f_{i,i}.$$

Then we have

Lemma 2.1[8] For a smooth function $f(t)$ defined on $R^1 \times S^{n-1}$, we have

$$f_1 = \frac{1}{|\gamma'|}f', \quad f_a = 0, \quad f_{11} = \frac{1}{|\gamma'|}(\frac{f'}{|\gamma'|})', \quad f_{aa} = \frac{\alpha' f'}{\alpha|\gamma'|^2}, \quad f_{ij} = 0 (i \neq j),$$

$$\Delta f = \frac{1}{|\gamma'|}(\frac{f'}{|\gamma'|})' + (n-1)\frac{\alpha' f'}{\alpha|\gamma'|^2} \tag{2.1.7}$$

On the following, we assume $(x^{\alpha})' = \alpha x^{\alpha-1}$ for all α, from the above we have

Lemma 2.2 A revolution hypersurface in R^{n+1} , defined by (2.1.1), is a nth order Willmore hypersurface if and only if φ satisfies the following equation:

$$\frac{1}{|\gamma'|^2}\sum_{k=1}^{n-1}(-1)^{k+1}C_{n-1}^{k}\sigma_1^{n-k-3}[(n-k-1)(n-k-2)\sigma_1'^2\sigma_k+2(n-k-1)\sigma_1\sigma_1'\sigma_k'$$
$$+(n-k-1)\sigma_1\sigma_1''\sigma_k+\sigma_1^2\sigma_k''+((n-1)\frac{\alpha'}{\alpha}-\frac{|\gamma'|'}{|\gamma'|})((n-k-1)\sigma_1\sigma_1'\sigma_k+\sigma_1^2\sigma_k')]$$
$$+\frac{1}{|\gamma'|^2}\sum_{k=2}^{n-1}(-1)^{k+1}(n-k)\sigma_1^{n-k-2}[x_0((n-k-1)\sigma_1'^2+\sigma_1\sigma_1''-\frac{|\gamma'|'}{|\gamma'|}\sigma_1\sigma_1')+(n-1)y_0\frac{\alpha'}{\alpha}\sigma_1\sigma_1']$$
$$+\sum_{k=2}^{n-1}(-1)^{k+1}\sigma_1^{n-k-1}C_{n-1}^{k}[n\sigma_1^2\sigma_k-(n-1)\sigma_2\sigma_k-\sigma_1\sigma_{k+1}]$$
$$+n(n-1)^2\sigma_1^{n-1}(\sigma_1^2-\sigma_2)=0, \tag{2.1.8}$$

where $x_0=\sum_{i=1}^{k}(-1)^{i-1}C_n^{k-i}\sigma_{k-i}\lambda^{i-1}, y_0=\sum_{i=1}^{k}(-1)^{i-1}C_n^{k-i}\sigma_{k-i}\mu^{i-1}$.

Proof Since $c = 0$, from (1.4) we have

$$\Delta(Q_{n-1}+\sigma_1^{n-1})+\sum_{k=2}^{n}(-1)^{k+1}T_{(k-1)ij}(\sigma_1^{n-k})_{,ij}+(n^2\sigma_1^2-n(n-1)\sigma_2)(Q_{n-1}+\sigma_1^{n-1})$$
$$-n\sigma_1Q_n+\sum_{k=2}^{n}(-1)^{k+1}C_n^k\sigma_1^{n-k}(n\sigma_1\sigma_k-(n-k)\sigma_{k+1})=0.$$

Noting

$$Q_{n-1}=\sum_{k=0}^{n-1}(-1)^{k+1}C_{n-1}^{k}\sigma_1^{n-k-1}\sigma_k,\ Q_n=\sum_{k=0}^{n}(-1)^{k+1}C_n^k\sigma_1^{n-k}\sigma_k,$$

and using Lemma 2.1 we have

$$\Delta(Q_{n-1}+\sigma_1^{n-1})=\Delta(\sum_{k=1}^{n-1}(-1)^{k+1}C_{n-1}^{k}\sigma_1^{n-k-1}\sigma_k)$$
$$=[\frac{1}{|\gamma'|}(\frac{1}{|\gamma'|}\sum_{k=1}^{n-1}(-1)^{k+1}C_{n-1}^{k}\sigma_1^{n-k-1}\sigma_k)']'+(n-1)\frac{\alpha'}{\alpha|\gamma'|^2}(\sum_{k=1}^{n-1}(-1)^{k+1}C_{n-1}^{k}\sigma_1^{n-k-1}\sigma_k)'$$
$$=\frac{1}{|\gamma'|^2}\sum_{k=1}^{n-1}(-1)^{k+1}C_{n-1}^{k}\sigma_1^{n-k-3}[(n-k-1)(n-k-2)\sigma_1'^2\sigma_k+2(n-k-1)\sigma_1\sigma_1'\sigma_k'$$
$$+(n-k-1)\sigma_1\sigma_1''\sigma_k+\sigma_1^2\sigma_k''+((n-1)\frac{\alpha'}{\alpha}-\frac{|\gamma'|'}{|\gamma'|})((n-k-1)\sigma_1\sigma_1'\sigma_k+\sigma_1^2\sigma_k')]. \tag{2.1.9}$$

From the definition of $T_{(k-1)}$ and Lemma 2.1 we have

$$T_{(k-1)ij}(\sigma_1^{n-k})_{,ij}$$
$$=(C_n^{k-1}\sigma_{k-1}-C_n^{k-2}\sigma_{k-2}h_{ii}+C_n^{k-3}\sigma_{k-3}h_{ii}^2+\cdots+(-1)^{k-1}h_{ii}^{k-1})(\sigma_1^{n-k})_{ii}$$
$$=x_0\frac{1}{|\gamma'|}(\frac{(\sigma_1^{n-k})'}{|\gamma'|})'+(n-1)y_0\frac{\alpha'}{\alpha}\frac{(\sigma_1^{n-k})'}{|\gamma'|^2}$$
$$=x_0\frac{1}{|\gamma'|}(\frac{(n-k)\sigma_1^{n-k-1}\sigma_1'}{|\gamma'|})'+(n-1)y_0\frac{\alpha'}{\alpha}\frac{(n-k)\sigma_1^{n-k-1}\sigma_1'}{|\gamma'|^2}$$
$$=\frac{1}{|\gamma'|^2}(n-k)\sigma_1^{n-k-2}[x_0((n-k-1)\sigma_1'^2+\sigma_1\sigma_1''-\frac{|\gamma'|'}{|\gamma'|}\sigma_1\sigma_1')+(n-1)y_0\frac{\alpha'}{\alpha}\sigma_1\sigma_1'], \tag{2.1.10}$$

where $x_0=\sum_{i=1}^{k}(-1)^{i-1}C_n^{k-i}\sigma_{k-i}\lambda^{i-1},\quad y_0=\sum_{i=1}^{k}(-1)^{i-1}C_n^{k-i}\sigma_{k-i}\mu^{i-1}$.

From (2.1.10) and (2.1.9) we have (2.1.8). This completes the proof of Lemma 2.2.

Lemma 2.3 A hypersurface φ defined by (2.1.1) is a nth order Willmore hypersurface if and only if φ satisfies the following equation :

$$\frac{1}{n^n|\gamma'|^2}\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k(\lambda+(n-1)\mu)^{n-k-3}\mu^{k-3}\{[k(n-k-1)(n-k+1)\lambda\mu^2 \tag{2.1.11}$$
$$+(n-k-1)(3k(n-1)+(n-k)(n-k-2))\mu^3]\lambda'^2+2[k(n(n-1)(n-k-1)$$
$$+(k-1)(n+k-1))\lambda\mu^2+k(k-1)(n-k)\lambda^2\mu(n-1)(n(n-1)(n-2)-k(k-1))\mu^3]\lambda'\mu'$$
$$+[k(k-1)(k-2)\lambda^3+k(k-1)(2(n-1)(n-3)+(n-k))\lambda^2\mu$$
$$+k(n-1)((n-1)(n-2)(n-3)+2(n-2)(n-k)+(n-1)(n-k-1))\lambda\mu^2$$
$$+(n-1)^3(k(n-k-1)+(n-2)(n-k))\mu^3]\mu'^2+[k(n-k+1)\lambda^2\mu^2+(k(n-1)(n-k+3)$$
$$+(n-k)(n-k-1))\lambda\mu^3+(n-1)(2k(n-1)+(n-k)(n-k-1))\mu^4]\lambda''$$
$$+[k(k-1)\lambda^3\mu+k((n-1)(n+k-2)+(n-k))\lambda^2\mu^2+(n-1)(kn(n-1)$$
$$+(n-k)(n+k-1))\lambda\mu^3+n(n-1)^3\mu^4]\mu''$$
$$+((n-1)\tfrac{\alpha'}{\alpha}-\tfrac{|\gamma'|'}{|\gamma'|})[(k(n-k)\lambda^2\mu^2+(k(n-1)(n-k+1)+(n-k)(n-k-1))\lambda\mu^3$$
$$+(n-1)(k(n-1)+(n-k)(n-k-1))\mu^4)\lambda'+(k(k-1)\lambda^3\mu+k((n-1)(n+k-3)+$$
$$(n-k))\lambda^2\mu^2+(n-1)(k(n-1)(n-2)+(n-k)(n+k-1))\lambda\mu^3+(n-1)^3(n-k)\mu^4)\mu']$$
$$+k((k-1)\tfrac{\alpha'}{\alpha}\lambda\mu+((n-k)\tfrac{\alpha'}{\alpha}-\tfrac{|\gamma'|'}{|\gamma'|})\mu^2)(\lambda+(n-1)\mu)^2(\lambda'+(n-1)\mu')\}$$
$$+\tfrac{1}{n^n|\gamma'|^2}n(n-1)^2(\lambda+(n-1)\mu)^{n-3}\{(n-2)\lambda'^2+2(n-1)(n-2)\lambda'\mu'+(n-1)^2(n-2)\mu'^2$$
$$+(n-1)(\lambda+(n-1)\mu)\lambda''+(n-1)(\lambda+(n-1)\mu)\mu''$$
$$+((n-1)\tfrac{\alpha'}{\alpha}-\tfrac{|\gamma'|'}{|\gamma'|})[(\lambda+(n-1)\mu)\lambda'+(n-1)(\lambda+(n-1)\mu)\mu']\}+F=0$$

where

$$F=\tfrac{1}{n^n}\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k(\lambda+(n-1)\mu)^{n-k-3}\mu^{k-3}[k\lambda^3+(n-2k-1)\lambda^2\mu$$
$$-(2n-k-2)\lambda\mu^2+(n-1)\mu^3](\lambda+(n-1)\mu)^2\mu^2+\tfrac{1}{n^n}(n-1)^2(\lambda+(n-1)\mu)^{n-1}(\lambda-\mu)^2.$$

Proof From the definition of σ_r we have:

$$\sigma_k=(C_n^k)^{-1}(C_{n-1}^{k-1}\lambda\mu^{k-1}+C_{n-1}^k\mu^k)=\tfrac{1}{n}\mu^{k-1}(k\lambda+(n-k)\mu),$$

$$\sigma_k'=\frac{k}{n}\mu^{k-2}(\lambda'\mu+(k-1)\lambda\mu'+(n-k)\mu\mu'),$$

$$\sigma_k''=\tfrac{k}{n}\mu^{k-3}[2(k-1)\lambda'\mu\mu'+(k-1)(k-2)\lambda\mu'^2$$
$$+(k-1)(n-k)\mu\mu'^2+\lambda''\mu^2+(k-1)\lambda\mu\mu''+(n-k)\mu^2\mu''],$$

$$\sigma_1^2=\frac{1}{n^2}(\lambda^2+2(n-1)\lambda\mu+(n-1)^2\mu^2),$$

$$\sigma_1'^2=\frac{1}{n^2}(\lambda'^2+2(n-1)\lambda'\mu'+(n-1)^2\mu'^2).$$

$$x_0=\sum_{i=1}^{k}(-1)^{i-1}C_n^{k-i}\sigma_{k-i}\lambda^{i-1}=\sum_{i=1}^{k}(-1)^{i-1}[C_{n-1}^{k-i-1}\lambda^i\mu^{k-i-1}+C_{n-1}^{k-i}\mu^{k-i}\lambda^{i-1}]$$
$$=C_{n-1}^{k-1}\mu^{k-1}=\tfrac{k}{n-k}C_{n-1}^k\mu^{k-1}.$$

Similarly, we have

$$y_0=\sum_{i=1}^{k}(-1)^{i-1}C_n^{k-i}\sigma_{k-i}\mu^{i-1}=C_{n-1}^k[\frac{k(k-1)}{(n-1)(n-k)}\lambda+\frac{k}{n-1}\mu]\mu^{k-2}.$$

By taking these into (2.1.8) we get (2.1.11). This completes the proof of Lemma 2.3.

2.2 ODE of the nth order Willmore revolution hypersurfaces

In the following we will simplify the equation (2.1.11). The key technic to simplify equation (2.1.11) is to choose a suitable parameter of curve γ. We need following Lemma:

Lemma 2.4[8] For a smooth curve $\gamma = (\alpha, \beta) : I \to R^2$, where α, β are smooth function on interval I, and $\alpha \neq 0$, we can choose a parameter t of γ, such that $|\gamma'|^2 = \alpha 2$.

In the following, we assume $\alpha > 0$, so γ and the rotational axis x^{n+1} have no intersection point. Thus we have $|\gamma'| = \alpha$, which implies $(\frac{\alpha'(t)}{\alpha(t)})^2 + (\frac{\beta'(t)}{\alpha(t)})^2 = 1$. So we can assume

$$\frac{\alpha'(t)}{\alpha(t)} = \frac{1}{\cosh y(t)},\ \frac{\beta'(t)}{\alpha(t)} = \tanh y(t), \tag{2.2.1}$$

where y is a function on R^1 . We see that $\gamma(t)$ is determined by function $y(t)$. We will derive the equation with respect to y from (2.1.11). From (2.1.6) and (2.2.1) we have

$$\lambda = \frac{y'}{\alpha \cosh y},\ \mu = \frac{\sinh y}{\alpha \cosh y}. \tag{2.2.2}$$

Furthermore, we have

$$\lambda' = \frac{y'' \cosh y - y'^2 \sinh y - y'}{\alpha \cosh^2 y},\ \mu' = \frac{y' - \sinh y}{\alpha \cosh^2 y}. \tag{2.2.3}$$

$$\lambda'' = \frac{y''' \cosh^2 y - 3y'y'' \sinh y \cosh y - y'^3(1-\sinh^2) - 2y'' \cosh y + 3y'^2 \sinh y + y'}{\alpha \cosh^3 y}, \tag{2.2.4}$$

$$\mu'' = \frac{y'' \cosh y - 2y'^2 \sinh y - y'(2-\sinh^2 y) + \sinh y}{\alpha \cosh^3 y}.$$

By a director but long computation we get the following theorem.

Theorem 2.1 *A hypersurface φ defined by* (2.1.1) *is a nth order Willmore hypersurface if and only if $y(t)$ satisfies the following ODE:*

$$\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k n^k (y' + (n-1)\sinh y)^{n-k-3} \sinh^{k-3} y\{k(n-k+1)\sinh^2 y \cosh^2 y y'^2 y'''$$
$$+(k(n-1)(n-k+3) + (n-k)(n-k-1))\sinh^3 y \cosh^2 y y' y''' + (n-1)(2k(n-1)$$
$$+(n-k)(n-k-1))\sinh^4 y \cosh^2 y y''' + k(n-k-1)(n-k+1)\sinh^2 y \cosh^2 y y' y''^2$$
$$+(n-k-1)(3k(n-1) + (n-k)(n-k-2))\sinh^3 y \cosh^2 y y''^2$$
$$+[2k(k-1)(n-k+1)\sinh y \cosh y - k(n-k+1)(2n-2k+1)\sinh^3 y \cosh y]y'^3 y''$$
$$+[(2k(k-1)(n+2k-2) + 2k(n-1)^2 + 2kn(n-1)(n-k-1) - 2k(n-k+1)(n-k)$$
$$+k(n-k-1))\sinh^2 y \cosh y - (3k(n-1)(3n-3k+1) + (n-k)(n-k-1)$$
$$(2n-2k-1))\sinh^4 y \cosh y]y'^2 y'' + [(2n(n-1)^2(n-2) + k(n-1)^2(n+k-1)$$
$$+(n-1)(n-k)(n+k-1) + k(n-1)(n-2)(n-k+1) - (n-k)(n-2k)(n-k-1)$$
$$-8k(n-1) - 2k(k-1)(2n+k-2) - 2k(n-1)(n+3)(n-k-1))\sinh^3 y \cosh y$$
$$-3(n-1)(2k(n-1) + (n-k)(n-k-1))\sinh^5 y \cosh y]y' y''$$
$$+[k(n-1)^2(2n-k-7) + (n-1)(n-4)(n-k)(n-k-1) - n(n-1)^2(n-3)$$
$$+2k(n-1)(k-1)]\sinh^4 y \cosh y y''$$
$$+k[(n-k)(n-k+1)\sinh^4 y - ((k-1)(2n-2k+3) + (n-k))\sinh^2 y + (k-1)(k-2)]y'^5$$
$$+[(n-k)(4k(n-1) + (n-k-1)^2)\sinh^5 y + (-kn(n-k) + k(n-k+1)(2n-2k+1)$$

$$
\begin{aligned}
&-2kn(n-1)(n-k-1)-k(k-1)(n+4k-2)-k(n-1)(3n+k-1)+k(n+2k-2)\\
&+(n-2k-1)-(n-k)(n-k-1))\sinh^3 y+k(k-1)(2(n-1)(n-3)\\
&+(n-k-2))\sinh y]y'^4+[(n-1)(2k(n-1)+(n-k)(n-k-1))\sinh^6 y\\
&+(k(n-1)(8n-6k+3)-2(k-1)(n-k)(n-k-1)+2k(k-1)(2n+k-2)\\
&-k(n-1)^2(2n+k)+2kn(n-1)(n-k-1)-2n(n-1)^2(n-2)+k(n-k)\\
&-2(n-1)(n-k)(n+k-1)-k(n-1)(n-2)(n-k+1)+2(n-1)(n-2k-1)\\
&-(2n-k-2))\sinh^4 y+(k(n-k-1)(2n-k-2)-k(n^2-1)(n-k-1)\\
&-2k(k-1)(n-1)(n-4)-k(k-1)(4n+k-4)+k(n-1)(n-2)(3n-k-3)\\
&+k(n-1)^2(n-2)(n-3)-2k(n-1)(n+k-2))\sinh^2 y]y'^3\\
&+[(-k(n-1)^2(n-k-9)-2n(n-1)^2-2k(n-1)(k-1)+(n-1)(n-k)(n+k-1)\\
&-(n-1)(n-5)(n-k)(n-k-1)+(n-1)^2(n-2k-1)-(n-1)(4n-2k-5))\sinh^5 y\\
&+((n-k)(n-k-1)^2+k(n-1)(4n-2k-1)+2k(n-1)^2(n-k-1)+2k^2(k-1)\\
&-2n(n-1)^2(n-2)+k(n-1)^2(n-2)(n+k-3)+(n-1)(n-2)(n-k)(n-k-2)\\
&+k(n-1)^3(n-k-1)+(n-1)^3(n-2)(n-k)-2k^2n(n-1)+k^2(n-k)\\
&-2k^2(n-1)(n-2)-2(n-1)(n-k)(n+k-1))\sinh^3 y]y'^2\\
&+[(k(n-1)^2+(n-1)^2(n^2-3n+4))\sinh^6 y+(-2n(n-1)^2+k(n-1)^2(k+3)\\
&-2(n-1)^2(n-2)(n-k)+k(n-1)^2(n-2)(n-k-3)-k(n-1)^3(2n-k-3)\\
&+2(n-1)^2(n-k)-(n-1)^3(n-2)(n-k)+2k(n-1)(n-2)(n-k)\\
&-2k(k-1)(n-1))\sinh^4 y]y'+n(n-1)^3\sinh^5 y+(n-1)^3\sinh^7 y\}\\
&+(n-1)^2(y'+(n-1)\sinh y)^{n-3}\{n(n-1)\cosh^2 yy'y'''+n(n-1)^2\sinh y\cosh^2 yy'''\\
&-n(5n-7)\sinh y\cosh yy'^2y''+[n(2n^2-8n+7)\cosh y-3n(n-1)^2\sinh^2 y\cosh y]y'y''\\
&-n(n-1)(2n-3)\sinh y\cosh yy''+(-n\sinh^2 y+(n^2-n+1))y'^4\\
&+[-n(n-1)^2\sinh^3 y+((n-2)(n+2)-n(n-1)(n-4)\sinh y)]y'^3\\
&+[((n-1)^2(2n+1)-4(n-1)+1)\sinh^2 y+n(n-1)^2(n-3)]y'^2\\
&+(n-1)[(n^2-3n+4)\sinh^3 y-n^2(n-2)]y'+(n-1)^2(\sinh^4 y+n\sinh^2 y)\}=0.
\end{aligned}
\tag{2.2.5}
$$

3. A classes of nth order Willmore hypersurfae in R^{n+1}

3.1 A classes of special solution of the equation

In this section, we concentrate on considering the solution of equation (2.2.5). It is very difficult to get all the solutions. Fortunately, we can get a classes of special solutions of this equation. In the fact, we can find the solutions of (2.2.5), satisfying

$$y' = A\sinh y + B\cosh y, \tag{3.1.1}$$

where A and B are some constants. For the purpose to determine constants A and B by using (2.2.5) and (3.1.1), we need a long computation. From (3.1.1) we have

$$y'' = 2AB\sinh^2 y + (A^2 + B^2)\sinh y\cosh y + AB,$$

$$y''' = 2A(A^2 + 3B^2)\sinh^3 y + 2B(3A^2 + B^2)\sinh^2 y\cosh y + A(A^2 + 5B^2)\sinh y + B(A^2 + B^2)\cosh y$$

Taking y', y'' and y''' into (2.2.5) and taking a long computation we have the following equation.

Lemma 3.1 *If y is a solution of equation (2.2.5) and (3.1.1) then we have*

$$\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^{k}n^{k}[(A+n-1)\sinh y+B\cosh y]^{n-k-3}\sinh^{k+3}y(P_{1k}\sinh y+P_{2k}\cosh y)$$
$$+(n-1)^{2}[(A+n-1)\sinh y+B\cosh y]^{n-1}[(A-1)^{2}+B^{2}]\sinh^{2}y$$
$$-n(n-1)^{2}P_{1n}[(A+n-1)\sinh y+B\cosh y]^{n-4}\sinh^{5}y$$
$$+2(n-1)^{2}[(A+n-1)\sinh y+B\cosh y]^{n-1}(A-1)B\sinh y\cosh y$$
$$-n(n-1)^{2}[(A+n-1)\sinh y+B\cosh y]^{n-4}P_{2n}\sinh^{4}y\cosh y$$
$$+\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^{k}n^{k}[(A+n-1)\sinh y+B\cosh y]^{n-k-3}\sinh^{k+1}y(P_{3k}\sinh y+P_{4k}\cosh y)$$
$$+(n-1)^{2}[(A+n-1)\sinh y+B\cosh y]^{n-1}B^{2}+n(n-1)[(A+n-1)\sinh y$$
$$+B\cosh y]^{n-4}[(n-1)P_{3n}\sinh^{3}y+P_{4n}\sinh^{2}y\cosh y]$$
$$+\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^{k}n^{k}[(A+n-1)\sinh y+B\cosh y]^{n-k-3}\sinh^{k-1}y(P_{5k}\sinh y+P_{6k}\cosh y)$$
$$+n(n-1)[(A+n-1)\sinh y+B\cosh y]^{n-4}(P_{5n}\sinh y+P_{6n}\cosh y)$$
$$+\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^{k}n^{k}[(A+n-1)\sinh y+B\cosh y]^{n-k-3}\sinh^{k-3}y$$
$$k(k-1)\{[(2n+3k-8)A+2n^{2}-7n+4-k]B^{4}\sinh y+(k-2)B^{5}\cosh y\}=0. \tag{3.1.2}$$

Where P_{1k}, P_{1n}, P_{2n}, P_{2k}, P_{3n}, P_{4n}, P_{3k}, P_{4k}, P_{5n}, P_{6n}, P_{5k} and P_{6k} are defined as follows:

$P_{1k} := -(A-1)(A+n-1)^{2}(kA+n-1)[(n-1)A+1]+\{(A+n-1)^{2}(3kA+n-2k-1)+(A-1)^{2}(kA+n-1)+2(A-1)(A+n-1)[k(3A-1)+2(n-1)]$

$-(n-k-1)[kA(A-1)(A+n-1)+(A+n-1)(A+n-2)(kA+n-k)$

$+(2(A-1)(A+n-1)+n-1)(k(A+n-1)+kA+n-k)]-k[A(A-1)(kA+n-k)+(A+n-1)^{2}((3k-2)A+n-2k)+2(A+n-1)((3k-1)A^{2}+2(n-2k)A-n+k))\ \ +(A+n-1)^{2}(A+n-2)+A(A-1)(A+n-1)+(A+n-1)(4(A-1)(A+n-1)+2n-2)]\}B^{2}+\{3kA+(n-2k-1)-2k(k-2)(A+n-1)-k(n-k)(A+n-2)-k((3k-2)A+(n-2k))\}B^{4}$,

$P_{1n}=A(A-1)(A+n-1)^{3}+(A+n-1)[A(A-1)+(A+n-1)(A+n-2)+4(A-1)(A+n-1)+2(n-1)]B^{2}+(A+n-2)B^{4}$,

$P_{2n}=[2A(A-1)(A+n-1)^{2}+2(A-1)(A+n-1)^{3}+(n-1)(A+n-1)^{2}]B+[2(A+n-1)(A+n-2)+2(A-1)(A+n-1)+(n-1)]B^{3}$,

$P_{2k}=\{-(n-k-1)[A(A-1)(A+n-1)(k(A+n-1)+kA+n-k)$

$+(A+n-1)(kA+n-k)(2(A-1)(A+n-1)+(n-1))]$

$-k[2A(A-1)(A+n-1)(kA+n-k)+(A+n-1)^{2}((3k-1)A^{2}+2(n-2k)A-(n-k))+2A(A-1)(A+n-1)^{2}+(A+n-1)^{2}(2(A-1)(A+n-1)+(n-1))]$

$+2(A-1)^{2}(A+n-1)(kA+n-1)+(A-1)(A+n-1)^{2}(k(3A-1)+2(n-1))\}B$

$+\{-(n-k-1)[(A+n-2)(k(A+n-1)+kA+n-k)+k(2(A-1)(A+n-1)+(n-1))]$

$-k(2(A+n-1)((3k-2)A+n-2k)$

$+(k-1)(A+n-1)^{2}+(3k-1)A^{2}+2(n-2k)A-(n-k)+2(A+n-1)(A+n-2)+2(A-1)(A+n-1)+(n-1))+2(A+n-1)(3kA+n-2k-1)+k(A+n-1)^{2}+(A-1)(k(3A-1)+2(n-1))\}B^{3}-k(k-2)B^{5}$,

$P_{3n}=(A-1)(A+n-1)^{3}((n-1)A-1)+(A+n-1)[(n-2)(A+n-2)(3A+n-2)+(n-2)(A-1)-(A+n-1)(A+n-2)-2(n-1)-(A-1)(2A+2n+7)]B^{2}+[(n-2)(A-1)(A+n-1)+2(A+n-2)]B^{4}$,

$P_{4n}=(A+n-1)^{2}[(n-2)(n-3)(A-1)^{2}+2(n-2)^{2}(A-1)(A+n-2)$

$+2(n-2)(A-1)(2A+n-3)+2(n-1)(A-1)^{2}+(n-1)(n-2)(3A+n-4)-(n-1)^{2}]B$

$+(n-1)[(n-2)(A+n-1)(A+n-2)+2(n-1)(A+n-1)$

$-(A-1)(A+n-1)+(A+n-2)-(n-1)]B^{3}$,

$P_{3k}=n(A-1)(A+n-1)^{2}(kA+n-1)((n-1)A-1)$

$+\{(n-k-1)(n-k-2)(A+n-2)((A+n-2)(kA+n-k)$

$+2k(A-1)(A+n-1))+2k(n-k-1)((k-1)(A-1)(A+n-1)^2$

$+(A+n-1)(2A+n-3)((2k-1)A+n-k)+(A-1)(A+n-2)(kA+n-k))$

$+(n-k-1)(-kA(A-1)(A+n-1)-(k(A+n-1)+kA+n-k)(2(A-1)(A+n-1)+(n-1))+k(A-1)^2(A+n-1)-(A+n-1)(A+n-2)(kA+n-k)$

$+((A-1)(A+n-1)-(A+n-2))(k(A+n-1)+kA+n-k))-k[A(A-1)(kA+n-k)+2(A+n-1)((3k-1)A^2+2(n-2\mathrm{k})A-(n-k))+(A+n-1)^2((3k^2-9k+5)A+k(n-3k)-(2n-5k))+k(A-1)^2(kA+n-k)+2(A+n-1)((3k^2-3k+1)A^2$

$+(2k-1)(n-3k)A-k(2n-3k))]+(n-2)(n-k-1)(k(A-1)(A+n-1)$

$+(A+n-2)(k(A+n-1)+kA+n-k))+k(n-2)((k-1)(A+n-1)^2$

$+(A-1)(kA+n-k)+2(A+n-1)((2k-1)A+n-2k))$

$+k((n-k-1)(A+n-1)(A+n-2)(3A+n-4)-A(A-1)(A+n-1)$

$-2(A+n-1)(2(A-1)(A+n-1)+n-1)-(A+n-1)^2(A+n-2)+(A-1)^2(A+n-1)+2(A+n-1)((A-1)(A+n-1)-(A+n-2))+(k-1)(A-1)(A+n-1)^3$

$+(A+n-1)(2A+n-3)((k-1)(A+n-1)+(k-1)A+(n-k))$

$+(A+n-1)(A+n-2)((k-1)A+n-k))+(A-1)^2(kA+n-1)$

$+2(A-1)(A+n-1)(k(3A-1)+2(n-1))+(A+n-1)^2(3kA+n-2k-1)\}B^2$

$+\{2k(k-2)(n-k-1)(A+n-2)+k(3k-2)A+k(n-2k)-2k(k-1)(A+n-1)+k((3k^2-9k+5)A+k(n-3k)-(2n-5k))+2k(k-1)(k-3)(A+n-1)+k(k-1)(n-2)-2k(A+n-2)+k(k-1)(A+n-1)(A+n-2)+k(3kA+n-2k-1)+2k(A+n-1)+(3kA+n-2k-1)\}B^4$,

$P_{4k}=\{(n-k-1)(n-k-2)(A-1)(A+n-1)[k(A-1)(A+n-1)+2(A+n-2)(kA+n-k)]+2k(n-k-1)(A-1)(A+n-1)((A+n-1)((2k-1)A+n-2k)+(2A+n-3)(kA+n-k))+(n-k-1)(A+n-1)((A-1)^2(k(A+n-1)+kA+n-k)+(A-1)$

$-(A+n-2)(kA+n-k))+k(A+n-1)(2k(A-1)^2(kA+n-k)+(A+n-1)(3k^2-3k+1)A+(2k-1)(n-3k)A-k(2n-3k))+(n-2)(n-k-1)(A+n-1)((A-1)(k(A+n-1)$

$+kA+n-k)+(A+n-2)(kA+n-k))+k(n-2)(A+n-1)(2(A-1)(kA+n-k)$

$+(A+n-1)((2k-1)A+n-2k))+k(A+n-1)^2((n-k-1)(A-1)^2+2(A-1)(A+n-2)+2(A-1)^2+(A-1)(A+n-1)-(A+n-2)+(A-1)((k-1)(A+n-1)+(k-1)A+n-k)+(2A+n-3)((k-1)A+n-k))\}B+\{(n-k-1)(n-k-2)(A+n-2)^2$

$+2k(n-k-1)[(k-1)(A+n-1)(2A+n-3)+(A+n-2)((2k-1)A+n-2k)]$

$-(n-k-1)[k(2(A-1)(A+n-1)+n-1)+(A+n-2)(k(A+n-1)+kA+n-k)$

$+k(A+n-2)]+k(-((3k-1)A^2+2(n-2k)A-(n-k))+2(A+n-1)((3k^2-9k+5)A+k(n-3k)-(2n-5k))+(k-1)(k-3)(A+n-1)^2+(3k^2-3k+1)A^2+(2k-1)(n-3k)A-k(2n-3\mathrm{k}))+k(n-2)[(n-k-1)(A+n-2)+2(k-1)(A+n-1)+(2k-1)A+n-2k]+k[(n-k-1)(A+n-2)^2-2(A-1)(A+n-1)-(n-1)-2(A+n-1)(A+n-2)+(A+n-1)^2((A-1)(A+n-1)-(A+n-2))+(k-1)(A+n-1)^2(2A+n-3)$

$+(A+n-2)((k-1)(A+n-1)+(k-1)A+n-k)]+(A-1)(k(3A-1)+2n-2)+2(A+n-1)(3kA+n-2k-1)\}B^3+k[(k-1)(k-4)+1]B^5$,

$P_{5n}=(A+n-1)(2(n-2)^2(A-1)(A+n-1)+(n-1)(n-2)(A+n-2)^2+2(n-2)(A-1)(A+n-2)+(n-1)(A-1)^2+2(n-1)(A-1)(A+n-1)$

$+(n-1)(n-2)(A-1)+2(n-1)(n-3)(A+n-2))B^2-(n-1)(A+n-2)B^4$,

$P_{6n}=(n-1)[(n-2)(A+n-1)(A+n-2)+(A-1)(A+n-1)-(A+n-2)]B^3$,

$P_{5k}=[(n-k-1)(n-k-2)(2k(A-1)(A+n-1)(A+n-2)+(A+n-2)^2(kA+n-k))+2k(n-k-1)((k-1)(A-1)(A+n-1)^2+(A+n-1)(2A+n-3)((2k-1)A+n-k)+(A-1)(A+n-2)(kA+n-k))+(n-k-1)(k(A-1)^2(A+n-1)+((A-1)(A+n-1)$

$-(A+n-2))(k(A+n-1)+kA+n-k))+2k(A+n-1)((3k^2-3k+1)A^2+(2k-1)(n-3k)-k(2n-3k))+k(k-1)(A+n-1)^2(3(k-1)A+n-3k)$

$+(n-2)(n-k-1)(k(A-1)(A+n-1)+(A+n-2)(k(A+n-1)+kA+n-k))+k(n-2)((A-1)(kA+n-k)+2(A$

$+ n - 1)((2k - 1)A + n - 2k) + (k - 1)(A + n - 1)^2)$

$+k(A + n - 1)(2(n - k - 1)(A - 1)(A + n - 2) + (n - k - 1)(A + n - 2) + (A - 1)^2$

$+2(A - 1)(A + n - 1) - 2(A + n - 2) + (k - 1)(A - 1)(A + n - 1)$

$+(2A + n - 3)((k - 1)(A + n - 1) + (k - 1)A + n - k) + (A + n - 2)((k - 1)A + n - k))]B^2$

$+[4k(k - 1)(n - k - 1)(A + n - 2) - k(n - k - 1)(A + n - 2)$

$+k(k(A - 1)2 (kA + n - k) + (3k^2 - 9k + 5)A + k(n - 3k) - (2n - 5k))$

$+2k(k - 1)(k - 3)(A + n - 1) + k(k - 1)(3(k - 1)A + n - 3k)$

$+2k(k - 1)(k - 2)(A + n - 1) + 2k(k - 1)(n - 2) - k(A + n - 2)$

$+k(k - 1)(A + n - 2) + k(k - 1)(A + n - 1)(A + n - 2) + 3kA + n - 2k - 1]B^4$,

$P_{6k} = [k(n - k - 1)((n - k - 2)(A + n - 2)^2 + 2k(k - 1)(A + n - 1)(2A + n - 3) +2k(A + n - 2)((2k - 1)A + n - 2k) + (A - 1)$
$(A + n - 1) - (A + n - 2)) +k((3k^2 - 3k + 1)A^2 + (2k - 1)(n - 3k)A - k(2n - 3k))$

$+2(k - 1)(A + n - 1)(3(k - 1)A + n - 3k) + (k - 1)(k - 2)(A + n - 1)^2$

$+k(n - 2)(n - k - 1)(A + n - 2) + 2k(k - 1)(n - 2)(A + n - 1) + k(n - 2)((2k - 1)A + n - 2k)$

$+k(n - k - 1)(A + n - 2) + k(A - 1)(A + n - 1) - k(A + n - 2)$

$+k(k - 1)(A + n - 1)(2A + n - 3) + k(A + n - 2)((k - 1)A + n - 1 + (k - 1)A + n - k)]B^3 +k(k - 1)(2k - 5)B^5$.

For the purpose to get the special solution of equation (3.1.2), firstly, we take $B = 0$. In this case, we have

$$P_{1k} = -(A - 1)(A + n - 1)^2 (kA + n - 1)[(n - 1)A + 1],$$
$$P_{1n} = A(A - 1)(A + n + 1)^3 ,\ P_{2n} = 0,\ P_{2k} = 0,$$
$$P_{3n} = (A - 1)(A + n - 1)^3\ [(n - 1)A - 1],\ P_{4n} = 0,$$
$$P_{3k} = n(A - 1)(A + n - 1)^2 (kn + n - 1)[(n - 1)A - 1],$$
$$P_{4k} = P_{5n} = P_{6n} = P_{5k} = P_{6k} = 0.$$

By putting these into (3.1.2) we have

$$\begin{aligned} &-\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k n^k(A+n-1)^{n-k-1}(A-1)(kA+n-1)[(n-1)A+1]\sinh^{n+1}y \\ &+(n-1)^2(A+n-1)^{n-1}(A-1)^2\sinh^{n+1}y - n(n-1)^2A(A-1)(A+n-1)^{n-1}\sinh^{n+1}y \\ &+\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k n^k n(A-1)(A+n-1)^{n-k-1}(kA+n-1)[(n-1)A-1]\sinh^{n-1}y \\ &+n(n-1)^2(A+n-1)^{n-1}(A-1)[(n-1)A-1]\sinh^{n-1}y = 0. \end{aligned} \tag{3.1.3}$$

Since 1, sinh y are liner independent we have

$$\begin{aligned} &\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k n^k(A+n-1)^{n-k-1}(A-1)(kA+n-1)[(n-1)A+1] \\ &+(n-1)^2(A+n-1)^{n-1}(A-1)[(n-1)A+1] = 0, \end{aligned} \tag{3.1.4}$$

and

$$\begin{aligned} &\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k n^k(A-1)(A+n-1)^{n-k-1}(kA+n-1)[(n-1)A-1] \\ &+(n-1)^2(A+n-1)^{n-1}(A-1)[(n-1)A-1] = 0. \end{aligned} \tag{3.1.5}$$

We see that (3.1.4) and (3.1.5) are the equations with respect to A. In the case of $n = 3$, we see that $A \neq -n + 1$ and we have $A = 1, -\frac{1}{2}$. In the following, we assume $n > 3$. It is evident that $A = 1$ is a solutions of equation (3.1.4) and (3.1.5). So we only need to consider the solutions with $A \neq 1$. As $A \neq 0$, $A \neq -(n - 1)$, it is easy to see that A is a solution of equations (3.1.4) and (3.1.5) if and only if it is the solution of following equation:

$$\begin{aligned} &\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k n^k(A+n-1)^{n-k-1}(kA+n-1) \\ &+(n-1)^2(A+n-1)^{n-1} = 0. \end{aligned} \tag{3.1.6}$$

Setting $x = n/(A + n - 1)$, we can write equation (3.1.6) as follows:

$$\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k x^k(kA+n-1)+(n-1)^2=0,$$

i.e.

$$(n-1)\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k x^k + Ax\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k kx^{k-1}+(n-1)^2=0. \tag{3.1.7}$$

Noting that for any real number t we have

$$\sum_{k=0}^{n-1}(-1)^{k+1}C_{n-1}^k t^k = -(1-t)^{n-1}.$$

So we have

$$\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k t^k = -(1-t)^{n-1}+1-(n-1)t.$$

As the derivatives of the functions on the both sides of above equation are equal, we have

$$\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k kt^{k-1} = (n-1)(1-t)^{n-2}-n+1.$$

For the same reason, we also have

$$\sum_{k=2}^{n-1}(-1)^{k+1}C_{n-1}^k k(k-1)t^{k-2} = -(n-1)(n-2)(1-t)^{n-3}.$$

Making use of these identities, we can write (3.1.8) as follows:

$$-(n-1)x-(1-x)^{n-1}+Ax[(1-x)^{n-2}-1]+n=0. \tag{3.1.9}$$

Taking $x = n/(A + n - 1)$ back to (3.1.9), we get

$$(A-1)^{n-2}((n-1)A+1)^2=0. \tag{3.1.10}$$

This shows A = 1, $-\frac{1}{n-1}$ are the solutions.

On the following, we considered $B \neq 0$, For the implicity of the equation,we only consider $n = 4$. We first simplify (2.1.11),and get the following equation :

$$\begin{aligned}&\tfrac{4}{|\gamma'|^2}\{[18\lambda'^2+12\lambda'\mu'+66\mu'^2+9(\lambda-\mu)\lambda''+3(\lambda-\mu)\mu''](\lambda-\mu)\\&+3(3\tfrac{\alpha'}{\alpha}-\tfrac{|\gamma'|'}{|\gamma'|})[(3\lambda^2+2\lambda\mu+11\mu^2)\lambda'+(\lambda^2+22\lambda\mu+25\mu^2)\mu']\\&+24[\tfrac{|\gamma'|'}{|\gamma'|}(\lambda+\mu)\mu-\tfrac{\alpha'}{\alpha}(\lambda^2+\lambda\mu+4\mu^2)](\lambda'+3\mu')\}\\&+[9(\lambda+3\mu)^3-48\mu(2\lambda^2+5\lambda\mu+5\mu^2)](\lambda-\mu)^2\end{aligned} \tag{3.1.11}$$

from (2.2.2) and (3.1.1) we have

$$\lambda'=\frac{(A-1)y'}{\alpha\cosh^2 y},\ \mu'=\frac{y'-\sinh y}{\alpha\cosh^2 y}. \tag{3.1.12}$$

$$\lambda''=\frac{(A-1)[(A-1)-y'\sinh y]y'}{\alpha\cosh^3 y} \tag{3.1.13}$$

$$\mu''=\frac{[-(y'-\sinh y)\sinh y+(A-1)]y'-(y'-\sinh y)}{\alpha\cosh^3 y} \tag{3.1.14}$$

By taking (2.2.2) , (3.1.1) , (3.1.12) , (3.1.13) , (3.1.14) into (3.1.11) and simplify it we get

$$[(3y'^3-9y'^2\sinh y+5y'\sinh^2 y+\sinh^3 y)(y'-\sinh y)-12(A-1)y'^2\sinh y(y'-\sinh y) \tag{3.1.15}$$
$$+24(A-1)^2y'^2+12A(A-1)(y'-\sinh y)y'-4(y'-\sinh y)^2](y'-\sinh y)=0$$

since $B \neq 0$,so $y' = A \sinh y + B \cosh y \neq \sinh y$, we get

$$(3y'^3-9y'^2\sinh y+5y'\sinh^2 y+\sinh^3 y)(y'-\sinh y)-12(A-1)y'^2\sinh y(y'-\sinh y) \tag{3.1.16}$$
$$+24(A-1)^2y'^2+12A(A-1)(y'-\sinh y)y'-4(y'-\sinh y)^2=0$$

then we take y′ = A sinh y + B cosh y into (3.1.16),we can get

$$[3B^4-2(9A^2-6A-1)B^2-(A-1)^2(3A+1)^2]\sinh^4 y-4(A-1)(6A^2-1)B\sinh^3 y\cosh y \tag{3.1.17}$$
$$+[6B^4+2(9A^2-24A+11)B^2+4(A-1)^2(9A^2-1)]\sinh^2 y$$
$$+4(A-1)(12A^2-15A-2)B\sinh y\cosh y+3B^4+4(9A^2-15A+5)B^2=0$$

Since$\{\sinh^4 y, \sinh^3 y \cosh y, \sinh^2 y, \sinh y \cosh y, 1\}$ are linear independence in any interval, so equation(3.1.17) is equivalent to:

$$\begin{cases} 3B^4-2(9A^2-6A-1)B^2-(A-1)^2(3A+1)^2=0 \\ (A-1)(6A^2-1)B=0 \\ 3B^4+(9A^2-24A+11)B^2+2(A-1)^2(9A^2-1)=0 \\ (A-1)(12A^2-15A-2)B=0 \\ 3B^4+4(9A^2-15A+5)B^2=0 \end{cases} \tag{3.1.18}$$

Solving the equations,we get the solution of (3.1.18):

$$A=1, B=\pm\sqrt{\frac{4}{3}} \tag{3.1.19}$$

3.2 A classes of nth order Willmore hypersurfae in R^{n+1}

Since the equation is very difficult, we can not get all the slotions, In the above section, we get two solutions of equation (3.1.2) satisfiys $y' = A \sinh y + B \cosh y$:

$$A=1, B=0; A=-\frac{1}{n-1}, B=0 \tag{3.2.1}$$

and when $n = 4$ we get

$$A=1, B=\pm\sqrt{\frac{4}{3}} \tag{3.2.2}$$

So, we have:

Theorem 3.1: Let S^{n-1} be the unit sphere in n−dimensional Euclidean space R^n , $\xi = (\xi 1 ,\cdots, \xi n)$ be the position vector of a point of S^{n-1} in R^n . $\gamma = (\alpha, \beta) : R^1 \to R^2$ be a smooth curve in R^2.

For the general $n$if the revolution hypersurface $\varphi(t, \xi) = (\alpha\xi, \beta)$ is nth order Willmore hypersurface. then the equation of γ is:

(i) $\alpha^2+(\beta-c_0)^2=c^2$;

(ii) $\beta=\pm a_0[\sum_{k=1}^{[\frac{n-1}{2}]}\frac{(n-1)(n-3)\cdots(n-2k+1)}{(n-2)(n-4)\cdots(n-2k)}(\frac{\alpha}{a_0})^{\frac{n-2k-1}{n-1}}\sqrt{(\frac{\alpha}{a_0})^{\frac{2}{n-1}}-1}$
$+\frac{1+(-1)^n}{2}\ln((\frac{\alpha}{a_0})^{\frac{1}{n-1}}+\sqrt{(\frac{\alpha}{a_0})^{\frac{2}{n-1}}-1})]+a_2.$

For $n = 4$ if the revolution hypersurface $\varphi(t, \xi) = (\alpha\xi, \beta)$ is 4th order Willmore hypersurface.then the equation of γ is:

(iii) $(\alpha\mp\sqrt{\frac{4}{3}}r)^2+(\beta-r_0)^2=r^2$;

Where $[\frac{n-1}{2}]$is take the integral of $\frac{n-1}{2}$. c_0 , c, r_0 , r , a_0 , a_1 , a_2 , c_1 are constant.

Proof : For general n,from (3.1.1) and (3.2.1) we get:

Case 1. $A = 1, B = 0.\ y' = \sinh y$.

From (2.2.2), (2.2.3), we have: $\lambda = \mu = c_1$ (*constant*), and φ is totally umbilical.

And from (2.2.1), (2.2.2) we have:

$$\frac{y'}{\alpha\cosh y} = c_1$$

$$\alpha = c\tanh y, c = \frac{1}{c_1}$$

$$\beta' = \alpha\tanh y = c\tanh^2 y$$

By integrating it, we have:

$$\beta = -c\frac{1}{\cosh y} + c_0(const)$$

We can write the equation of γ as:

$$\alpha^2 + (\beta - c_0)^2 = c^2 \tag{3.2.3}$$

Case 2. $A = -\frac{1}{n-1}$, $B = 0.\ y' = -\frac{1}{n-1}\sinh y$.

Integrating sinh $\frac{y'}{\sinh y} = -\frac{1}{n-1}$ on both side, we have :

$$-\frac{t}{n-1} + a_1 = \int\frac{y'}{\sinh y}dt = \int\frac{dy}{\sinh y} = \int\frac{d\frac{y}{2}}{\sinh\frac{y}{2}\cosh\frac{y}{2}} = \int\frac{d\frac{y}{2}}{\tanh\frac{y}{2}\cosh^2\frac{y}{2}}$$
$$= \ln\tanh\frac{y}{2} = \ln\frac{\cosh y-1}{\sinh y} = \ln\frac{\cosh y-1}{\sqrt{\cosh^2 y-1}} = \frac{1}{2}\ln\frac{\cosh y-1}{\cosh y+1}$$

And:

$$\frac{\cosh y - 1}{\cosh y + 1} = e^{2a_1-\frac{2t}{n-1}}, \cosh y = \frac{1 + e^{2a_1-\frac{2t}{n-1}}}{1 - e^{2a_1-\frac{2t}{n-1}}} = \frac{1}{\tanh(\frac{t}{n-1} - a_1)}$$

So

$$\tanh y = \frac{\sinh y}{\cosh y} = \pm\frac{\sqrt{\cosh^2 y-1}}{\cosh y}$$
$$= \pm\sqrt{1 - \tanh^2(\frac{t}{n-1} - a_1)} = \pm\frac{1}{\cosh(\frac{t}{n-1}-a_1)}$$

From the first equation of (2.2.1) we get:

$$\frac{\alpha'(t)}{\alpha(t)} = \tanh(\frac{t}{n-1} - a_1)$$

By integrating it we get:

$$\alpha(t) = a_0\cosh^{n-1}(\frac{t}{n-1} - a_1) = a_0\cosh^{n-1} u$$

From the second equation of (2.2.1) ,we have :

$$\beta'(t) = \alpha(t)\tanh y = \pm a_0\cosh^{n-2}(\frac{t}{n-1} - a_1)$$

By itegrating it we have:

$$\beta(t) = \pm a_0[\sum_{k=1}^{[\frac{n-1}{2}]}\frac{(n-1)(n-3)\cdots(n-2k+1)}{(n-2)(n-4)\cdots(n-2k)}\cosh^{(n-2k-1)} u\sinh u + \frac{1+(-1)^n}{2}u] + a_2$$

from $\beta'(t) = \alpha(t)\tanh y = \pm a_0\cosh^{n-2}(\frac{t}{n-1} - a_1)$ we get :

$$\cosh u=(\frac{\alpha(t)}{a_0})^{\frac{1}{n-1}},\ \sinh u=\pm(\cosh^2 u-1)^{\frac{1}{2}}=\pm((\frac{\alpha(t)}{a_0})^{\frac{2}{n-1}}-1)^{\frac{1}{2}}.$$

$$u=\pm\ln((\frac{\alpha}{a_0})^{\frac{1}{n-1}}+\sqrt{(\frac{\alpha}{a_0})^{\frac{2}{n-1}}-1}).$$

So the equation of γ can be written as :

$$\beta=\pm a_0[\sum_{k=1}^{[\frac{n-1}{2}]}\frac{(n-1)(n-3)\cdots(n-2k+1)}{(n-2)(n-4)\cdots(n-2k)}(\frac{\alpha}{a_0})^{\frac{n-2k-1}{n-1}}\sqrt{(\frac{\alpha}{a_0})^{\frac{2}{n-1}}-1}$$
$$+\frac{1+(-1)^n}{2}\ln((\frac{\alpha}{a_0})^{\frac{1}{n-1}}+\sqrt{(\frac{\alpha}{a_0})^{\frac{2}{n-1}}-1})]+a_2 \quad (3.2.4)$$

Where $u=\frac{t}{n-1}-a_1$, $[\frac{n-1}{2}]$ is take the integral of $\frac{n-1}{2}$. a_0 , a_1 , a_2 , c_0 , c, r_0 , r are constants.

Case 3. For $n=4$,from (3.1.1) and (3.2.1) we get:

$$A=1, B=\pm\sqrt{\frac{4}{3}}.\quad y'=\sinh y\pm\sqrt{\frac{4}{3}}\cosh y$$

By calculating the first equation of (2.2.3), we have $\lambda'=0$, and $\lambda=constant$, write $\frac{1}{r}$. From the first equation of (2.2.2) we have:

$$\frac{\sinh y\pm\sqrt{\frac{4}{3}}\cosh y}{\alpha\cosh y}=\frac{1}{r}$$

So

$$\alpha=r\tanh y\pm\sqrt{\frac{4}{3}}r$$

Furthermore, from the second equation of (2.2.1) and the first equation of (2.2.2) we have:

$$\beta'=ry'\frac{\tanh y}{\cosh y}=ry'\frac{\sinh y}{\cosh^2 y}$$

By integrating it we get:

$$\beta=-r\frac{1}{\cosh y}+r_0(const)$$

So the equation of γ is:

$$(\alpha\mp\sqrt{\frac{4}{3}}r)^2+(\beta-r_0)^2=r^2 \quad (3.2.5)$$

Let $\alpha(t)(\xi_1, \xi_2, \cdots, \xi_n)=(x_1, x_2, \cdots, x_n)$, $\alpha\xi_i=x_i$, $\beta(t)=x_{n+1}$.
$\xi:=\{(\xi_1, \xi_2, \cdots, \xi_n)\in R^n \mid \sum\xi_i^2=1\}$,$(1\le i\le n)$. then $\varphi=(\alpha(t)\xi, \beta(t))=(x_1, x_2, \cdots, x_{n+1})$.

Theorem 3.1': For general n, The following hypersurfaces in R^{n+1} are nth order Willmore hypersurface :

(i) $S^n=\{(x_1,x_2,\cdots,x_{n+1})\in R^{n+1}\mid\sum x_i^2+(x_{n+1}-c_0)^2=c^2\}$;

(ii) $U^n=\{(x_1,x_2,\cdots,x_{n+1})\in R^{n+1}\mid$

$$x_{n+1}=\pm a_0[\sum_{k=1}^{[\frac{n-1}{2}]}\frac{(n-1)(n-3)\cdots(n-2k+1)}{(n-2)(n-4)\cdots(n-2k)}(\frac{1}{a_0}\sqrt{\sum x_i^2})^{\frac{(n-2k-1)}{n-1}}((\frac{1}{a_0}\sqrt{\sum x_i^2})^{\frac{2}{n-1}}-1)^{\frac{1}{2}}$$
$$+\frac{1+(-1)^n}{2}\ln((\frac{1}{a_0}\sqrt{\sum x_i^2})^{\frac{1}{n-1}}+((\frac{1}{a_0}\sqrt{\sum x_i^2})^{\frac{2}{n-1}}-1)^{\frac{1}{2}})]+a_2\}.$$

(iii) For $n=4$, $T^4=\{(x_1,x_2,x_3,x_4,x_5)\in R^5\mid(\sqrt{\sum x_i^2}\mp\sqrt{\frac{4}{3}}r)^2+(x_5-r_0)^2=r^2\}$;

is 4th order Willmore hypersurface

Where $[\frac{n-1}{2}]$ is take the intgral, c_0 , c, r_0 , r , a_0 , a_1 , a_2 , c_1 are constants. And any hypersurface which is conformal equivalent to S^n , U^n or T^4 is nth order Willmore or 4th Willmore.

Remark: (i) When $n = 3$, Theorem 3.1'is Theorem 3.1 in [8].

(ii) when n is odd, β is an algebraic fanction of α; When n is even β is an transcendental fanction of α . This is a well thing.

(iii) In this paper we only get a classes of special W_n –minimal hypersurfaces in Euclidean space, Wo can also concider other W_n –minimal hypersurfaces in Euclidean space.

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