



Finite-Dimensional Hopf Algebras

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Abstract: This paper mainly discussed various characterizations for the finite-dimensional Hopf algebras over algebraically closed field and has characteristic 0. And further showed that the order of antipode of the Hopf algebras is finite, but also provides a hint on how to estimate the order of the antipodes.

Keywords: Finite-dimensional Hopf algebras; Order of the antipode; Trace; Semisimple; Cosemisimple; Eigenvalue; Distinguished grouplike

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1. Introduction

Throughout this paper K is a algebraically closed field and has characteristic 0, H is a finite -dimensional $K - Hopf$ algebra with antipode S which is a diagonalizable operator and C is a $K -$

coalgebra. There is a convenient adaptation of the Heyneman-Sweedler^[1] sigma notation for coalgebras and comodules as $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ and $\rho(c) = \sum c_{(-1)} \otimes c_{(0)} \quad \forall c \in C$.

Definition 1.1^[1] A grouplike elements of C is a $c \in C$ which satisfies the following conditions: $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$, the set of grouplike elements of C is denoted $G(C)$.

We firstly recall the following actions as module structures:

H^* is a left $H -$ module via $(h \rightarrow h^*)(g) = h^*(gh)$ for $h, g \in H, h^* \in H^*$.

H^* is a right $H -$ module via $(h \leftarrow h^*)(g) = h^*(hg)$ for $h, g \in H, h^* \in H^*$.

H is a left $hH^* -$ module via $h^* \rightarrow h = \sum h^*(h_2)h_1$ for $h^* \in H^*, h \in H$.

H is a right $H^* -$ module via $h \leftarrow h^* = \sum h^*(h_1)h_2$ for $h^* \in H^*, h \in H$.

If $g \in H$ is a grouplike element as in Definition 1.1, we can denote by

$$L_g = \{m \in H^* \mid h^*m = h^*(g)m \text{ for any } h^* \in H^*\}$$

and

$$R_g = \{n \in H^* \mid nh^* = h^*(g)n \text{ for any } h^* \in H^*\}$$

which are ideals of H^* and $L_1 = \int_l, R_1 = \int_r$. Also recall from [1] that L_g and R_g are 1-dimensional, and there exists a grouplike element d such that $R_d = L_1$, where d is called the distinguished grouplike. We can perform the same constructions on the dual algebra H^* . More precisely, for any $\eta \in G(H^*) = \text{Alg}(H, K)$ we can define

$$L_\eta = \{x \in H \mid hx = \eta(h)x \text{ for any } h \in H\}$$

$$R_\eta = \{y \in H \mid yh = \eta(h)y \text{ for any } h \in H\}.$$

We remark that if we keep the same definition we gave for L_g , then L_η should be a subspace

of H^{**} . The set L_η , as defined above, is just the preimage of this subspace via the canonical

isomorphism $\theta: H \rightarrow H^{**}$. From the above it follows that the subspaces L_η and R_η are ideals of dimension 1 in H , and there exists $\alpha \in G(H^*)$ such that $R_\alpha = L_\varepsilon$. This element α is the distinguished grouplike element in H^* .

Remark 1.2^[1] If H is semisimple and cosemisimple, then distinguished grouplike in H and

H^* are equal to 1 and ε , respectively.

Lemma 1.3^[2] Suppose that H is a Hopf algebra over K .

Then

The only subspaces of H which are both a left ideal and left coideal of H are $\{0\}$ and H .

If H contains a non-zero finite -dimensional left or right ideal. Then H is finite-dimensional.

Lemma 1.4^[3] Let C be a finite-dimensional coalgebra over K . Then $U \mapsto U^\perp$ is a one-one inclusion reversing correspondence between the set of coideals (respectively subcoalgebras, left coideals, right coideals) of C and the set of subalgebras (respectively ideals, left ideals, right ideals) of the dual algebra C^* .

Lemma 1.5^[3] If $C_n(K)$ is a simple coalgebra over K for all $n \geq 1$. Then any simple coalgebra over K is isomorphic to $C_n(K)$ for some $n \geq 1$.

Lemma 1.6^[4] Suppose U and V be vector spaces over K and $F: V^* \rightarrow U^*$ is the transpose of a linear map $f: U \rightarrow V$. If J and I are subspaces of V^* and U^* respectively. Then

$$F(J) \subseteq I \text{ implies } f(I^\perp) \subseteq J^\perp.$$

Remark 1.7^[4] For a subspace V of U let $res_V^U: U^* \rightarrow V^*$ be the restriction map which is thus defined by $res_V^U(u^*) = u^*|_V$ for all $u^* \in U^*$. Notice that $Ker(res_V^U) = V^\perp$. Hence

$U^*/V^\perp \cong V^*$ as vector spaces. Therefore we have the formula $Dim(U^*/V^\perp) = Dim(V^*)$. In particular V^\perp is a cofinite subspace of U^* if and only if V is a finite-dimensional subspace of

U . Also notice that $res_V^U = i^*$, where $i: V \rightarrow U$ is the inclusion map.

Definition 1.8^[4] For $a \in H$, $a^* \in H^*$, $b \in H$, define endomorphisms $L(a^*)$ and $R(a^*)$ in $End(H)$ by $L(a^*)(b) = a^* \rightarrow b$ and $R(a^*)(b) = b \leftarrow a^*$, on the other hand, $l(a)$ and $r(a)$ in $End(H)$ by $l(a)(b) = ab$ and $r(a)(b) = ba$.

Proposition 1.9^[5] Suppose that S is the antipode of H . Let Λ be a left integral for H and ω be a right integral for H^* which satisfy $\langle \Lambda, \omega \rangle = 1$. Then

$$Tr(r(a) \circ S^2 \circ R(a^*)) = \langle \omega, a \rangle \langle a^*, \Lambda \rangle \text{ for all } a \in H, a^* \in H^*.$$

The functional $\omega_r \in H^*$ defined by $\omega_r(a) = Tr(r(a) \circ S^2)$ for all $a \in H$ is a right integral for H^* .

Proposition 1.10^[5] Suppose that S is the antipode of H . Then the following are equivalent:

H and H^* are semisimple.

$$Tr(S^2) \neq 0.$$

Proposition 1.11^[5] Suppose that S is the antipode of H .

Let g and α be the distinguished grouplike elements for H and H^* respectively. Then $S^4 = \tau_g \circ (\tau_{\alpha^{-1}})^*$

or equivalently, $S^4(a) = g(\alpha \rightarrow a \leftarrow \alpha^{-1})g^{-1}$, for all $a \in H$.

If H and H^* are unimodular, in particular if H and H^*

are semisimple, then $S^4 = 1_H$.

$$Tr(S^2) = (Dim(H))Tr(S^2|_{x_H} H).$$

Theorem 1.12^[6] Let H be a Hopf algebra over K . Then the following are equivalent:

All left H - comodules are completely reducible.

$$\langle \lambda, 1 \rangle \neq 0 \text{ for some } \lambda \in \int^r.$$

$H = K1 \otimes C$ for some subcoalgebra C of H .

$$\langle \lambda, 1 \rangle \neq 0 \text{ for some } \lambda \in \int^l.$$

All right H - comodules are completely reducible.

Theorem 1.13^[6] Let H be a cosemisimple Hopf algebra with antipode S . Then $S^2(C) = C$ for all simple subcoalgebras C of H .

2. The order of the antipode

Lemma 2.1 Suppose $\eta \in G(H^*)$, $g \in G(H)$, $m, n \in H^*$ and $x \in L_\eta$ such that $m \rightarrow x = x \leftarrow n$. Then $m \in L_g$ and $n \in R_g$.

Proof Let $h^*, g^* \in H^*$. Then

$$\begin{aligned} (g^* h^* m)(x) &= \sum (g^* h^*)(x_1 m(x_2)) \\ &= (g^* h^*)(g) \\ &= g^*(g) h^*(g) \\ &= \sum g^*(m(x_2) x_1) h^*(g) \\ &= (g^* h^*(g) m)(x) \end{aligned}$$

which shows that $(g^*(h^* m - h^*(g) m))(x) = 0$, so $(h^* m - h^*(g) m)(x \leftarrow H^*) = 0$. But

$x \leftarrow H^* = H$, since $L_\eta \leftarrow \eta = L_\epsilon$ and $L_\epsilon \leftarrow H^* = H$ (applied for the dual of H^{op}). This

shows that $h^* m = h^*(g) m$, and so $m \in L_g$. The fact that $n \in R_g$ is proved in a similar way.

Corollary 2.2 If $m \in H^*$, $x \in L_\epsilon$, and $m \rightarrow x = 1$, then $m \in L_1$ and $x \leftarrow m = d$.

Proof If $h^* \in H^*$, then

$$\begin{aligned} h^*(x \leftarrow m) &= \sum h^*(x_2) m(x_1) \\ &= (m h^*)(x) \\ &= h(d) m(x) \\ &= h^*(m(x) d) \end{aligned}$$

Applying \mathcal{E} to the relation $\sum m(x_2) x_1 = 1$ we get $m(x) = 1$. This shows that $x \leftarrow m = d$. The fact that $m \in L_1$ is proved by Lemma 2.1.

Lemma 2.3 Suppose $x \in L_\eta$, $g \in G(H)$, $m \in H^*$ such that $m \rightarrow x = g$. Then for any

$h^* \in H^*$ we have $\eta(g)h^*(1) = \sum h^*(x_1)m(gx_2)$.

Proof From the fact that $\Delta(h^*) = \sum h_1^* \otimes h_2^*$,

$g = m \rightarrow x$ and $\eta(g)x = gx$, we have

$$\begin{aligned} \eta(g)h^*(1) &= \sum \eta(g)h_1^*(g^{-1})h_2^*(g) \\ &= h_1^*(g^{-1})h_2^*(m(x_2)x_1\eta(g)) \\ &= h_1^*(g^{-1})h_2^*(m(gx_2)gx_1) \\ &= \sum h^*(g^{-1}gx_1)m(gx_2) \\ &= \sum h^*(x_1)m(gx_2) \end{aligned}$$

Lemma 2.4 Let $x \in L_\eta$, $g \in G(H)$, $m \in H^*$, and $\eta \in G(H^*)$ such that $m \rightarrow x = g$. Then

for any $h \in H$ we have

$$S(g^{-1}(\eta \rightarrow h)) = (m \leftarrow h) \rightarrow x.$$

Proof If $h^* \in H^*$. Then

$$\begin{aligned} h^*(S(g^{-1}(\eta \rightarrow h))) &= \sum h^*(S(h_1)g)\eta(h_2) \\ &= \sum \eta(g)h^*(S(h_1)g)\eta(g^{-1}h_2) \\ &= \eta(g)((h^*S)\eta)(g^{-1}h) \\ &= \sum ((h_1^*S)\eta)(g^{-1}h)\eta(g)h_2^*(1) \\ &= \sum ((h_1^*S)\eta)(g^{-1}h)h_2^*(m(gx_2)x_1) \\ &= \sum (h_1^*S)(g^{-1}h_1)\eta(g^{-1}h_2)h_2^*(m(gx_2)x_1) \\ &= \sum h_1^*(S(h_1)g)h_2^*(m(gg^{-1}h_3x_2)g^{-1}h_2x_1) \\ &= \sum h_1^*(S(h_1)gg^{-1}h_2x_1m(h_3x_2)) \\ &= \sum h^*(x_1m(hx_2)) \\ &= h^*((m \leftarrow h) \rightarrow x). \end{aligned}$$

Remark 2.5 If we write the formula from Lemma 2.4 for the Hopf algebras $H, H^{cop}, H^{op,cop}$ and H^{op} , we get that for any $h \in H$ the following relations hold:

Suppose $x \in L_\eta, m \rightarrow x = g$, then

$$S(g^{-1}(\eta \rightarrow h)) = (m \leftarrow h) \rightarrow x;$$

Suppose $x \in R_\eta, m \rightarrow x = g$, then

$$S^{-1}((\eta \rightarrow h)g^{-1}) = (h \rightarrow m) \rightarrow x;$$

Suppose $x \in R_\eta, x \leftarrow n = g$, then

$$S((h \leftarrow \eta)g^{-1}) = x \leftarrow (h \rightarrow n);$$

Suppose $x \in L_\eta, x \leftarrow n = g$, then

$$S^{-1}(g^{-1}(h \leftarrow \eta)) = x \leftarrow (n \leftarrow h).$$

In particular

If $x \in L_\varepsilon, m \rightarrow x = 1$, then $S(h) = (m \leftarrow h) \rightarrow x$ (2.1)

If $x \in R_\alpha = L_\varepsilon, m \rightarrow x = 1$, then $S^{-1}(\alpha \rightarrow h) = (h \rightarrow m) \rightarrow x$ (2.2)

If $x \in R_\alpha = L_\varepsilon, x \leftarrow n = d$, then

$$S((h \leftarrow \alpha)d^{-1}) = x \leftarrow (h \rightarrow n) \quad (2.3)$$

If $x \in L_\varepsilon, x \leftarrow n = g$, then

$$S^{-1}(g^{-1}h) = x \leftarrow (n \leftarrow h) \quad (2.4)$$

Theorem 2.6 For any $h \in H$ we have

$$S^4(h) = d^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})d.$$

Proof Suppose $x \in L_\varepsilon = R_\alpha$, and $m \in H^*$ with $m \rightarrow x = 1$. Corollary 2.2 shows that $m \in L_1$ and $m \in L_1$ and $x \leftarrow m = d$. Moreover, we have

$$(S^4(h) \rightarrow m) \rightarrow x = S^{-1}(\alpha \rightarrow S^4(h))$$

(by (2.2))

$$= S^{-1}(S^4(\alpha \rightarrow h))$$

$$= S(S^2(\alpha \rightarrow h))$$

$$= (m \leftarrow S^2(\alpha \rightarrow h)) \rightarrow x \quad (\text{by (2.1)})$$

Since the map from H^* to H , sending $h^* \in H^*$ to $h^* \rightarrow x \in H$ is bijective, we obtain

$$S^4(h) \rightarrow m = m \leftarrow S^2(\alpha \rightarrow h).$$

On the other hand,

$$x \leftarrow (m \leftarrow S^2(\alpha \rightarrow h)) = S^{-1}(d^{-1}S^2(\alpha \rightarrow h))$$

(by (2.4))

$$= S^{-1}(S^2(d^{-1}(\alpha \rightarrow h)))$$

$$= S(d^{-1}(\alpha \rightarrow h))$$

$$= S(d^{-1}(\alpha \rightarrow h)dd^{-1})$$

$$= S(((d^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})d) \leftarrow \alpha)d^{-1})$$

$$= x \leftarrow ((d^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})d) \rightarrow m)$$

(by (2.3))

Since the map $h^* \mapsto (x \leftarrow h^*)$ from H^* to H is bijective, we obtain that

$$m \leftarrow S^2(\alpha \rightarrow h) = (d^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})d) \rightarrow m$$

We got that

$$S^4(h) \rightarrow m = (d^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})d) \rightarrow m$$

then the formula follows from the bijectivity of the map

$h \mapsto (h \rightarrow m)$ from H to H^* .

Theorem 2.7 Let H be a finite dimensional Hopf algebra. Then the antipode S has finite order.

Proof By Theorem 2.6, we obtain by induction that

$$S^{4n}(h) = d^{-n}(\alpha^n \rightarrow h \leftarrow \alpha^{-n})d^n$$

for any positive integer n . Since $G(H)$ and $G(H^*)$ are finite groups, their elements have finite orders, so there exists p for which $d^p = 1$ and $\alpha^p = \varepsilon$. Then it follows that $S^{4p} = I$.

3 Characterizations of semisimple Hopf algebras

Semisimple Hopf algebras are finite-dimensional by part (2) of Lemma 1.3. We characterize finite-dimensional Hopf algebras which are semisimple in the algebraically closed characteristic zero case. To this end we calculate a trace.

Lemma 3.1 If C is a simple coalgebra over K , and T is a diagonalizable coalgebra automorphism of C . The

$$Tr(T) = (\sum_{i=1}^n \lambda_i)(\sum_{i=1}^n \lambda_i^{-1})$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues for T .

Proof By lemma 1.5 we obtain that $C \cong C_n(K)$ for some $n \geq 1$. Thus we may assume

$C = C_n(K)$. The crux of the proof will be to show that there is a simple left coideal M of C

such that $T(M) \subseteq M$. Necessarily $Dim(M) = n$.

T^* is an algebra automorphism of $C^* = M_n(K)$. By Skolem-Noether Theorem, there is an invertible matrix $u \in M_n(K)$ such that $T^*(a) = uau^{-1}$ for all $a \in M_n(K)$. Identify $C^* = M_n(K)$ with $End(V)$, where V is an n -dimensional vector space over K . Since K is algebraically closed, u has an eigenvalue $\lambda \in K$. Let $v \in V$ be a non-zero vector satisfying $u(V) = \lambda v$. Regard $End(V)$ and V as left $End(V)$ -modules via function composition and evaluation respectively. Then V is a simple module and the evaluation map $e_v : End(V) \rightarrow V$ given by $e_v(a) = a(v)$ for all $a \in End(V)$

is a module map. Therefore $L = Ker(e_v) = \{a \in End(V) | a(v) = 0\}$ is a maximal left ideal of

$End(V)$ of codimension $n^2 - n$. Observe that $T^*(L) \subseteq L$. Set $M = L^\perp$. Then M is a minimal left coideal of C by Lemma 1.4 and $T(M) \subseteq M$ by Lemma 1.6. and Using Remark 1.7 we see that $Dim(M) = n$.

Since T is diagonalizable and $T(M) \subseteq M$ it follows that the restriction $T|_M$ is diagonalizable. Let $\{m_1, m_2, \dots, m_n\}$ be a basis of eigenvectors for $T|_M$ and let $\lambda_1, \dots, \lambda_n \in K$ satisfy $T(m_i) = \lambda_i m_i$ for all $1 \leq i \leq n$. Then $\lambda_1, \dots, \lambda_n$ are non-zero scalars since $T|_M$ is non-

zero. For each $1 \leq i \leq n$ write $\Delta(m_i) = \sum_{j=1}^n c_{i,j} \otimes m_j$. Then the $c_{i,j}$'s satisfy the comatrix identities and thus span a non-zero subcoalgebra D of C . Since C is simple $D = C$. Since

$Dim(C) = n^2$ necessarily the $c_{i,j}$'s form a basis for C . Applying $T \otimes T$ to both sides of the

equation for $\Delta(m_i)$ yields $\sum_{j=1}^n \lambda_i c_{i,j} \otimes m_j = \sum_{j=1}^n T(c_{i,j}) \otimes \lambda_j m_j$. Therefore $T(c_{i,j}) = \lambda_j \lambda_i^{-1} c_{i,j}$ for all $1 \leq i, j \leq n$. Since $\{c_{i,j}\}_{1 \leq i, j \leq n}$ is a basis for C we calculate

$$Tr(T) = \sum_{i,j=1}^n \lambda_i \lambda_j^{-1} = (\sum_{i=1}^n \lambda_i) (\sum_{i=1}^n \lambda_i^{-1})$$

Theorem 3.2 Let H be a Hopf algebra with antipode S over K . Then the following are equivalent.

H is cosemisimple.

$$Tr(S^2) \neq 0.$$

H is semisimple.

$$S^2 = 1_H.$$

$\omega : H \rightarrow K$ defined by $\omega(a) = Tr(r(a))$ for all $a \in H$ is a right integral for H .

Proof (1) \Rightarrow (2). Since H is cosemisimple it is the direct sum of its simple subcoalgebras. Let C be a simple subcoalgebra of H . Then $S(C) = C$ By Theorem 1.13. Now S^2 has finite order by part (1) of Theorem Proposition 1.11. Since K is algebraically closed of

characteristic zero S^2 is diagonalizable. Thus $Tr(S^2) = (\sum_{i=1}^n \lambda_i) (\sum_{i=1}^n \lambda_i^{-1})$ where $\lambda_1, \dots, \lambda_n$ are roots

of unity by Lemma 3.1. Since the characteristic of K is zero we may assume that $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, the field of complex numbers. Thus

$$Tr(S^2|_C) = (\sum_{i=1}^n \lambda_i) (\sum_{i=1}^n \lambda_i^{-1}) = (\sum_{i=1}^n \lambda_i) (\overline{\sum_{i=1}^n \lambda_i}) = \left| \sum_{i=1}^n \lambda_i \right|^2$$

is a non-negative real number. Therefore $Tr(S^2) = 1 + \sum_C Tr(S^2|_C) \geq 1$, where C runs over the simple subcoalgebras $C \neq K1$ of H . We have shown that $Tr(S^2) \neq 0$.

(2) \Rightarrow (3). It is pretty obvious by Proposition 1.10.

(3) \Rightarrow (4). Assume that H is semisimple. Then H^* is cosemisimple. We have just show H^*

is semisimple; thus H is semisimple and cosemisimple. In particular $Tr(S^2) \neq 0$. Now $Tr(S^2) = (Dim(H) Tr(S^2|_{x_H} H))$ by part (3) of Proposition 1.11 and $S^4 = 1_H$ by part (2) of Proposition 1.11. Since the characteristic of K is not 2, the last equation implies S^2 is a diagonalizable endomorphism of H with eigenvalues ± 1 . Choose a basis of eigenvectors for S^2 . Let n_+ be the number of basis vectors belonging to the eigenvalue 1 and let n_- be the number belonging to -1. By the preceding trace formula $n_+ - n_- = (n_+ + n_-)m$ for some integer m which is not zero since $Tr(S^2) \neq 0$. Squaring both sides of this equation yields $-2n_+n_- = (m^2 - 1)n_+^2 + 2m^2n_+n_- + (m^2 - 1)n_-^2 \geq 0$.

Therefore $n_+n_- = 0$. Since $n_+ \neq 0$ necessarily $n_- = 0$. We have shown $S^2 = 1_H$.

(4) \Rightarrow (5). That it is very simple follows by part (2) of Proposition 1.9.

(5) \Rightarrow (1). Since $\omega(1) = Dim(H)1 \neq 0$, thus our proof is complete by Theorem 1.12.

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