European Option Pricing under the Structural Time Series and Markov Regime-switching Model

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Abstract: We present closed-form formulas for the valuation of a European call option whose underlying process is assumed to follow structural time series and the Markov regime-switching process through mean reversion described by a harmonic oscillator. In our model, each parameter has related corresponding economic meaning, and this leads to an easy analy-sis of the interplay between the option and business cycles. Forward rates are assumed under the Heath et al. (1992) HJM framework. The call option analytic formulas are obtained when the joint distribution of occupation times is specified and forward rates are restricted in a one-factor HJM model.

Keywords: Structural time series; Markov regime-switching process; HJM; European call option

1. Introduction

A European option valuation can be regarded as the basis of derivatives valuation. Following the studies of Black and Scholes (1973) and Merton (1973), the option literature has developed into a crucial area of research. However, evidence illustrates that option pricing formulas such as the Black–Scholes (BS) model do not perform well empirically; the poor empirical performance of the BS model is attributed to its asymmetric leptokurtic feature and volatility smile. Various attempts have been executed to manage this apparent failure of the BS model. One method for modifying the BS formula involves generalizing the geometric Brownian motion that is used as a model for the dynamics of log stock prices or stochastic interest rates incorporated into an option pricing model.

Because asset–return distributions exhibit heavier tails, leptokurtic features, and volatility smiles, several researchers have modified asset–return distribution settings from statistical methods, such as assumptions of stochastic volatility and jump risk. Kou (2002), Glasserman and Kou (2003), and Hsu and Chen (2012) report jumps in the paths of interest rates and asset returns. Incorporating jump risk into a model can indicate that asset–return distributions exhibit heavier tails and leptokurtic properties.

Heston and Nandi (2000) and Duan (1995) develop a closed-form option valuation formula for a spot asset whose variance follows the GARCH (p,q) process. Kim and Kim (2004) investigates the improvement in the pricing of Korean KOSPI 200 index options when stochastic volatility is considered, and they compare the empirical performance levels of four classes of stochastic volatility option pricing models. Another assumption of stochastic volatility is the Markov regime-switching (MS) model, such as in Janczura and Weron (2010). Chen *et al*. (2014) consider the valuation of European quanto call options whose underlying process is driven by a regime-switching jump-diffusion model with Markov-modulated Poisson processes.

Hamilton (1989) develops an MS model for GDP that assumes that the growth rate follows a nonlinear stationary process and is subjected to discrete shifts in regimes, where the regimes are discrete episodes over which the dynamic behavior of the series is markedly different. The MS model not only provides improved statistical revision but also prov-

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ides economic intuition for the underlying process. This implies that the corresponding option valuation has more economic meaning. Therefore, option valuation can offer essential information for portfolio managers and hedge strategy managers.

On the basis of the economic intuition to select an appropriate model, we choose the structured time series (STS) model with mean revision for this study. The STS model (Harvey, 1993) is constructed in terms of components that have a direct interpretation, because it is regarded as the sum of trend, seasonal, and irregular components; a cycle is also incorporated. The STS model can not only provide forecasts but also present a set of stylized facts (Jones (2016); Jones and Tarp (2017)). Several technical developments and applications of the STS model (Koopman and Lucas, 2005; Akar and Baskaya, 2011) are available, and a summary of the main structured model and its properties can be found in Harvey (1993). In addition, asset returns are shown in mean-revision style in certain financial markets. To combine the cycle-trend component of the STS model and mean-revision behavior, Moreno and Platania (2011) assume that the mean-reversion level is modeled using a harmonic oscillator; they use this new model to price a zero-coupon bond. Laine (2019) shows how various useful models can be constructed for analyzing trends and other sources of variability in geodetic time series. Galati *et al*. (2016) extract the financial cycles of the United States and the five largest Euro area countries over the period 1970–2014 using the STS model.

The STS model detailed in the present paper is expressed in harmonic-oscillator form to facilitate the determination of the intuitive and economic relationship between the model's parameters and various financial markets. This is an advantage for valuing European call options. In addition, we extend the approach of Moreno and Platania (2011) because assuming that the underlying process depends on the business cycle appears reasonable. The underlying process is reduced to a lower level in economic recessions, whereas the long-term level tends to increase during economic expansions. However, this approach does not totally exploit business cycle characteristics. As mentioned, the asset prices behave differently in different regimes. Therefore, model parameters are assumed to be generated by hidden Markov motions (HMM). To combine statistical and economic perspectives, we price a European option call under the STS and MS (STS-MS) models with mean reversion described by a harmonic oscillator.

We simulate the underlying process and European call option prices using the STS-MS model. We also analyze the sensitivity of European call option prices regarding changes in business cycle patterns, underlying volatility, and long-term equilibrium level. The results of our simulation conducted using the STS-MS model verify the advantage of our model.

If the stochastic interest rate is incorporated into the pricing model, the accuracy of the option valuation is enhanced. Hence, stochastic interest rates are also considered in our model. The term structure of interest rates is modeled after the Heath *et al*. (1992) HJM model to easily extend it to multiple factors.

The contributions of this paper include the compilation of statistics that were previously lacking, inclusion of interest rate risk, discussion of the effect of business-cycle patterns on the evaluation of European call options, and provision of information for investors and hedgers who adopt investment or hedging strategies for European call options in various financial markets under different business-cycle patterns.

After deriving the closed-form formula of a call option under the STS-MS model with mean reversion, we restrict the forward rate to follow the one-factor HJM model. We then obtain the analytic solution for a European call option. The remainder of this paper is organized as follows. In Section 2, we construct an STS-MS model with mean reversion and two transition probability matrices. In Section 3, relevant mathematics and probability concepts are presented. In Section 4, closed-form formulas for the vanilla European call option are presented under the STS-MS model. In Section 5, the simulation and sensitivity analysis are presented, and finally, Section 6 details the conclusions.

2. Model framework

The description of regime-switching space is as follows: A continuous-time financial market is considered with a finite time horizon $[0,T]=\Upsilon$, where $T<\infty$ under a complete probability space $(\Omega,\Im,\mathrm{P})$. Define $\mathbf{X}=\{X(t)\}_{t\in\Upsilon}$, ($X(t)\in\mathbb{R}^N$) as a Markov chain; $\mathbf{\Pi}$ is a transition probability matrix for $\mathbf{X}$. Denote the elements of $\mathbf{\Pi}$ as Π_{ij}; that is, $\mathbf{\Pi}=[\Pi_{ij}(t)]_{i,j=1,2,\ldots,N}$. The transition matrix is constructed so that each entry is a transition function $\Pi_{ij}(s,t)=\Pr\left\{X_{s+t}=j\middle|X_s=i\right\}$.

Denote asset price and interest rate as $S(t)$ and $r(t)$, respectively. An STS and STS-MS model with a mean-reversion property, regime shift, and cyclical trend is described as follows:

$$dy(t)=\kappa(\theta(t)-y(t))dt+\sigma(t)dw_S(t) \quad , (1)$$

$$\theta(t)=\alpha\sin(\omega t)+\beta(t) \quad ;(2)$$

where $y(t)=\log S(t)$; κ represents the mean-revision speed of y; and θ represents the long-term equilibrium level, which follows a time-dependent cyclic process according to a harmonic oscillator in which α, ω, and β are the semiamplitude, temporal frequency, and intercept of the wave, respectively. Moreover, $w_S(t)$ represents a standard Brownian motion for asset prices y. In our model, let $\{\sigma(t)\}_{t\in\Upsilon}$ and $\{\beta(t)\}_{t\in\Upsilon}$ be modulated by a common continuous time; a finite-state Markov chain is given as follows:

$$\sigma(t)=\left\langle\boldsymbol{\sigma},X^V(t)\right\rangle,\boldsymbol{\sigma}\in\mathbb{R}^N, \quad \beta(t)=\left\langle\boldsymbol{\beta},X^g(t)\right\rangle,\boldsymbol{\beta}\in\mathbb{R}^N \quad , (3)$$

where $\langle\cdot,\cdot\rangle$ denotes an inner product; X^V and X^g denote the transition probability matrices of volatility and growth rate, respectively.

The characteristics of stock returns documented in the relevant literature involve a negative correlation between stock returns and volatility (Scott, 1997). However, errors are easily caused in estimated parameters if high growth rate and low growth rate regimes are determined by solely using the volatility value. McConnell and Perez-Quiros (2000) and Buckle *et al*. (2002) propose a model in which the low growth phase of the business cycle is a regime with two states and the high phase of the volatility cycle is a regime with two states. As noted by Buckle *et al*. (2002), “Such a hierarchical classification of time scales is one of the features of hidden Markov models and provides a relatively simple and open structure on which to build an overall model for asset price path.” This methodology is advantageous in the STS-MS model because the STS model comprises trend, seasonal, and irregular components. When the individual transition probability matrix is given a one-to-one individual component, the estimated parameters directly reflect the economic target.

3. Prior knowledge

The pricing formulas of a European call option are obtained under a constant interest rate and stochastic interest rate. First, for example, we explain how to price a European call option under a HMM and a constant interest rate.

3.1 Pricing call option under an HMM

Let $\Im^i=\left\{\Im^i(t)\right\}_{t\in\Upsilon}$ and $i=X,\ S,r$ denote the right-continuous, complete filtration generated by the Markov chain of an asset and asset price processes, respectively. Define $\mathcal{G}=\Im^X\vee\Im^S$ as an enlarged σ-algebra generated by $\Im^X$ and $\Im^S$ for each $t\in\Upsilon$. Suppose that two regimes are involved. Consider a European call option C on a stock with strike price K and maturity T. Under the constant interest rate and the HMM (under incomplete information), the payoff of $C(t)$ can be expressed in terms of occupation times given by

$$C(t)=e^{-r(T-t)}E^Q\left[C(T)\middle|\Im^S(t)\right]=E^Q\left[e^{-r(T-t)}E^Q\left[C(T)\middle|\mathcal{G}^S(t)\right]\middle|\Im^S(t)\right]$$

$$= E^{Q}\left[C_{Tra}(t;X^{S})\middle|\Im^{S}(t)\right] = \int_{t}^{T^{*}} C_{\ker nel}(t;O'_{t,T})\upsilon(x,\tau)dx \,,\quad \tau = T - t \quad (4)$$

where $C_{\ker nel}(t;O'_{t,T}) = S(t)N(d'_1) - K\exp(-r\tau)N(d'_2)$; (5)

$$d'_1 = \frac{\log(\frac{S(t)}{K}) + r\tau + \frac{1}{2}O'_{t,T}}{\sqrt{O'_{t,T}}} \,;\quad d'_2 = d'_1 - \sqrt{O'_{t,T}} \,;\quad O'_{t,T} = \sum_{i=1}^{2}\sigma_{S,i}^{2}J_i^{S}(t,T) \quad .(6)$$

$\sigma_{S,i}$ represents the volatility of the asset price in regime i. $C_{Tra}(t;X^S)$ in Eq. (4) is the well-known BS formula under complete information. Through the observation of Eq. (5), the traditional pricing formula expressed in the state view is transformed into the occupation–time view of the BS formula. The state call price $C_{Tra}(t;X^S)$ in Eq. (4) can be regarded as the vanilla call option price under the conditional filtration $\mathcal{G}^S(t)$ (including filtration $\Im^X$). The kernel call price $C_{\ker nel}(t;O'_{t,T})$ can be derived using the BS formula in which the adjusted parameters $O'_{t,T}$ are multiplied by the corresponding state's occupation time. The adjusted parameters in Eq. (6) are those modulated by the Markov chain. In the preceding illustration, the stock price volatility depends on the states. Therefore, the adjusted parameters $O'_{t,T}$ should be computed as the volatility square multiplied by the corresponding state's occupation time. If the joint distribution of occupation times $\upsilon(x,\tau)$ is specified, derivatives are priced using the kernel price and adjusted parameters under the underlying process generated by the HMM. This method facilitates the mathematical tractability of pricing derivatives under the underlying process generated using the HMM. Therefore, our objective is to develop a kernel price formula (in a traditional expression) and adjusted parameters.

3.2 Change of measure under the HMM

Because the market described by the HMM is incomplete, more than one equivalent martingale measure exists. Gerber and Shiu (1994) pioneer the use of the Esscher transform to value options in an incomplete market. Liew and Siu (2010) reveal that the Esscher transform and the extended Girsanov principle under the HMM lead to the same pricing result. Therefore, we use the extended Girsanov principle under the HMM (Liew and Siu, 2010) to determine the risk-neutral measure Q, which is given as follows:

Let $\lambda(t)$ denote the market price of risk that is assumed constant on each regime and each $t \in \Upsilon' \setminus \{0\}$, where Υ' is a time index. Write $\phi(x)$ for the probability density function of a standard normal distribution. Consider a $\mathcal{G}^S-$ adapted process $\xi = \left\{\xi(t)\middle| t \in \Upsilon' \setminus \{0\}\right\}$ given by:

$$\xi(t) = \frac{\phi(w_S(t) + \lambda(t))}{\phi(w_S(t))} \,,\quad t \in \Upsilon' \setminus \{0\} \,,\quad \xi(0) = 1 \,,\quad \mathrm{P} - a.s.$$

Define a process $\Lambda = \left\{\Lambda(t)\middle| t \in \Upsilon' \setminus \{0\}\right\}$ where

$$\Lambda(t) = \begin{cases} \prod_{k=1}^{t}\xi(t), & t \in \Upsilon' \setminus \{0\} \\ 1, & t = 0 \end{cases} .$$

Therefore, Λ is a $\mathcal{G}^S-$ adapted process and a $(\mathcal{G}^S, \mathrm{P})$ -martingale. Define a new probability measure $Q \sim \mathrm{P}$ on $\mathcal{G}_T^S$ by using the extended Girsanov principle by defining $\left.\frac{dQ}{d\mathrm{P}}\right|_{\mathcal{G}_T^S} = \Lambda_T$.

Under the risk-neutral measure Q, Eq. (1) can be expressed by

$$dy(t) = \kappa(\theta(t) - \frac{\lambda(t)\sigma(t)}{\kappa} - y(t))dt + \sigma(t)dw_S^Q(t) \quad (7)$$

The market price of risk $\lambda(t)$ depends on the states under the STS-MS model. According to Moreno and Platania (2011), we have

$$y(s) = y(t)e^{-\kappa(s-t)} + \alpha\frac{\kappa^2}{\kappa^2 - \omega^2}H(t,s) + (\beta(t) - \frac{\lambda(t)\sigma(t)}{\kappa})\left[1 - e^{-\kappa(s-t)}\right] + \sigma(t)\int_t^s e^{-\kappa(s-u)}dw_S{}^Q(u) \quad (8)$$

where $H(t,s) = \sin(\omega s) - e^{-\kappa(s-t)}\sin(\omega t) - \frac{\omega}{\kappa}\left(\cos(\omega s) - e^{-\kappa(s-t)}\cos(\omega t)\right)$.

3.3 Pricing a call option under forward measure

In this subsection, we introduce the concept of change of measure, which is required for the discussion on pricing call options. Define a Radon–Nikodym derivative Q_T for maturity T as $\frac{dQ_T}{dQ} = \frac{P(T,T)/B(T)}{P(0,T)/B(0)}$, where $B(t)$ is the money market account. By using the measure Q, we obtain the present value of a European call option determined using

$$C(t) = E^Q\left[\exp(-\int_t^T r(s)ds)(S(T)-K)^+ \middle| \mathcal{G}_t\right]$$

$$= P(t,T)E^{Q_T}\left[(S(T)-K)^+ \middle| \mathcal{G}_t\right] = P(t,T)\pi[K, \chi_S^{Q_T}],$$

where $\pi[\cdot]$ represents the payoff function and $\chi_S^{Q_T}$ represents the characteristic function under the measure Q_T. Given the Q-martingale stochastic process w_S^Q, $\chi_S^{Q_T}$ can be derived as follows:

$$\chi_S^{Q_T} = E^Q\left[\frac{dQ_T}{dQ}\exp(iuw_S^Q(t))\right] = E^Q\left[\frac{P(T,T)/B(T)}{P(0,T)/B(0)}\exp(iuw_S^Q(t))\right].$$

Note that the definition of the characteristic function is set in a natural measure. These notations are adopted because they are conveniently represented pricing under the forward measure.

4. Pricing a call option under the STS-MS model

4.1 Pricing a kernel of a call option under the STS-MS model

First, our objective is to develop a kernel price formula (in a traditional expression) and adjusted parameters for a European call option under the STS-MS model. We consider two conditions: constant interest rate and stochastic interest rate.

4.1.1 Condition 1: Constant interest rate

Under the constant interest rate and the HMM, the kernel price of a call option is given by

$$C_{Tra}^{case1}(t; X^S) = e^{-r(t)\tau}E^Q\left[(S(T)-K)^+ \middle| \Im^S(t)\right].$$

Proposition 1. Consider a vanilla call price with a payoff $w(S_T)$ at time T and the constant interest rate r under the STS-MS model. Let $v(x) = w(e^{-x})$ and assume that $x \mapsto e^{-Mx}|v(x)|$ is bound and integrable for some $M \in$, such that the moment generating function (mgf) of $w_S(T)$ satisfies $mgf(-M) < \infty$. At the current time

t, the traditional call price $C^{case1}_{Tra}(t;X^S)$ is:

$$C^{case1}_{Tra}(t;X^S)=S(t)e^{-r\tau+\varepsilon}\frac{e^{hM}}{2\pi}\int_{-\infty}^{\infty}e^{ihu}\frac{\chi_S^Q(iM\ -u)}{(M\ +iu)(M\ +iu+1)}du \quad (9)$$

where $\varepsilon=\dfrac{K}{S(t)}$, (10)

$$h=\varepsilon-\left\{y(t)e^{-\kappa(T-t)}+\alpha\frac{\kappa^2}{\kappa^2-\omega^2}H\ (t,T)+(\beta(t)-\frac{\lambda(t)\sigma(t)}{\kappa})\left[1-e^{-\kappa(T-t)}\right]\right\}, \quad (11)$$

$$\chi_S^Q(u)=E\left[\exp(iu\cdot\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u))\right]. \quad (12)$$

Proof: See Appendix A.

Given the characteristic function of S, the term χ_S^Q in Eq. (12) can be obtained.

4.1.2 Condition 2: Stochastic interest rate

The stochastic differential equation for the instantaneous forward rate $f(t,T)$ under the one-factor HJM model is obtained using

$$df(t,T)=\mu_f(t,T)dt+\sigma_f(t,T)dw_f(t) \quad (13)$$

where $\mu_f(t,T)$ is the drift function and $w_f(t)$ is a standard Wiener process. The relative correlation between $dw_S(t)$ and $dw_f(t)$ is ρdt.

Under the risk-neutral measure Q, the forward rate $f(t,T)$, the spot rate $r(t)$, and ZCB price processes can be expressed as follows:

$$df(t,T)=\xi_1(t,T)dt+\sigma_f(t,T)dw_f^Q(t)$$

$$r(t)=f(0,t)+\int_0^t\xi_1(s,t)ds+\int_0^t\sigma_f(s,t)dw_f^Q(s)$$

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left\{-\left[\int_t^T\int_0^t\xi_1(s,u)dsdu+\int_t^T\int_0^t\sigma_f(s,u)dw_f^Q(s)dsdu\right]\right\},$$

where $\xi_1(t,T)=\sigma_f(t,T)\int_t^T\sigma_f(t,s)ds=\sigma_f(t,T)\xi_2(t,T)$ and $\xi_2(t,T)=\int_t^T\sigma_f(t,s)ds$.

Set $\dfrac{\partial\sigma_f(t,T)}{\partial T}=-\varsigma(T)\sigma_f(t,T)$, for some deterministic function $\varsigma(T)$.

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left\{-\frac{1}{2}\beta^2(t,T)\phi(t)+\beta(t,T)\left[f(0,t)-r(t)\right]\right\},0\le t\le T \quad (14)$$

where $\beta(t,T)=\int_t^T e^{-\int_t^u\varsigma(v)dv}ds,t\le T$ and $\phi(t)=\int_0^t\sigma_f^2(s,t)ds$.

Under the one-factor HJM model, the kernel price formula of a call option is given by

$$C^{case2}_{Tra}(t;X^S)=E^Q\left[\exp(-\int_t^T r(s)ds)(S(T)-K)^+\left|\Im^S(t)\vee\Im^r(t)\right.\right]$$

$$=P(t,T)E^{Q_T}\left[(S(T)-K)^+\left|\Im^S(t)\vee\Im^r(t)\right.\right]=P(t,T)\pi[K,\chi_S^{Q_T}].$$

Proposition 2. Consider a vanilla call price with a payoff $w(S_T)$ at time T under the one-factor HJM model and

STS-MS model. The restriction of M follows Proposition 1. At the current time t, the traditional call price $C_{Tra}^{case2}(t;X^S)$ is

$$C_{Tra}^{case2}(t;X^S)=P(t,T)S(t)e^{\varepsilon}\frac{e^{hM}}{2\pi}\int_{-\infty}^{\infty}e^{ihu}\frac{\chi_S^{Q_T}(iM-u)}{(M+iu)(M+iu+1)}du,$$

$$\chi_S^{Q_T}=\exp\left(\int_0^T\left[-\frac{1}{2}\xi_2(s,T)^2\right]ds\right)\cdot\Psi(t),$$

where ε, h and $P(t,T)$ follow Eqs. (10), (11) and (14), and

$$\Psi(t)=E^Q\left[\exp(\int_0^T-\xi_2(s,T)dW_f^Q(s))\exp(iu\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S^Q(u))\right]. \quad (15)$$

Proof: See Appendix B.

Given the characteristic function of S and $f(t,T)$, the term $\Psi(t)$ in Eq. (15) can be obtained.

Pricing a call option under the STS-MS model

Propositions 1 and 2 contain the sets of the adjusted parameters $O_{t,T}^{case1}$ and $O_{t,T}^{case2}$, including $(\lambda(t),\sigma(t),\beta(t))$ involved in $C_{Tra}^{case1}(t;X^S)$ and $C_{Tra}^{case2}(t;X^S)$. The kernel call price $C_{\mathrm{ker}nel}^{case1}(t;O_{t,T}^{case1})$ and $C_{\mathrm{ker}nel}^{case2}(t;O_{t,T}^{case2})$ can be derived using the traditional pricing formula in which the adjusted parameters $O_{t,T}^{case1}$ and $O_{t,T}^{case2}$ are multiplied by the corresponding state's occupation time.

Under the STS-MS model, $C^{case\ i}(t)$, $i=1,2$, is given by

$$C^{case\ i}(t)=\int_t^T\int_t^T C_{\mathrm{ker}nel}^{case\ i}(t;O_{t,T}^{case\ i})\upsilon_V(x,\tau)\upsilon_g(z,\tau)dxdz,$$

where the kernel price $C_{\mathrm{ker}nel}^{case\ i}(t;O_{t,T}^{case\ i})$ is defined in Propositions 1 and 2. $\upsilon_V(x,\tau)$ and $\upsilon_g(z,\tau)$ represent the joint distribution of occupation time for volatility and growth rate, respectively. Given the joint distribution of occupation time for volatility and growth rate, the closed-form formulas of a European call option can be obtained under the STS-MS model.

5. Numerical analysis

We employ Proposition 5 to simulate European call option prices using the STS-MS model. We also analyze the sensitivity of European call option prices under changes in business cycle patterns, underlying volatility, and long-term equilibrium level.

We assume that the existence of macroeconomic shifts that cause the asset value to change between two states, namely e_1 (the good state) and e_2 (the bad state), of a hidden Markov chain. The corresponding transition probability matrix of the two-state Markov chain X is determined using $\mathbf{\Pi}=\begin{pmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{pmatrix}$. In addition, the transition probability matrix is assumed by the symmetric matrix; that is, $\Pi_{11}=\Pi_{22}$. According to Chen *et al*. (2019), the larger Π_{11} is, the longer the stay is; this implies that a larger Π_{11} corresponds to a longer business cycle. We focus on the relationship between the business cycle and call option price. Furthermore, we refer to the setting of the transition intensity matrix in McKinlay (2009). The following matrices are employed to represent the two types of economy: that with a long-term cycle (Type 1) and that with a short-term cycle (Type 2). The transition matrix, $R=\begin{bmatrix}-\lambda & \lambda\\ \nu & -\nu\end{bmatrix}$, is the generator of the

Markov chain such that equation (3.8) in McKinlay (2009) holds and $\lambda = \nu$ in the symmetric matrix. We assume that $\lambda = 1$ for Type 1 and $\lambda = 2$ for Type 2 because if λ increases (decreases), Π_{11} decreases (increases) (Pedler, 1971).

In a harmonic oscillator, $\sigma(t)$ and $\beta(t)$ partly depend on a Markov chain. Therefore, we assume that $\sigma(t)$ and $\beta(t)$ have different levels in each regime. For example, $\beta\big|_{e_1}$ = 1.1 and $\beta\big|_{e_2}$ = 2.2; $\sigma\big|_{e_1}$ = 0.25 and $\sigma\big|_{e_2}$ = 0.55. To simplify the analysis, the other parameters are set to the same value and the interest rate is kept constant. The parameters are set as follows: $(\kappa, \alpha, \omega, r)$ = (0.8, 0.005/3, 0.025, 0.05).

5.1 Simulation of stock price under the STS-MS model

First, the stock price path is simulated using Monte Carlo simulation in the STS-MS model. By observing the stock price path, we can obtain the behaviors underlying the different economies. We can then understand the relationship between economical meaning and the parameter setting and infer the call price's performance in the STS-MS model. This highlights why call prices are valued in the STS-MS model.

Consider the stock price process without the MS effect; this is called the STS model.[1] Volatility has the Markov switching (MS) effect, and this is called the STS-MS-OnV model. [2] Long-term equilibrium level has the MS effect, and this is called the STS-MS-OnB model. [3] The STS model is regarded as the benchmark model. The baseline model is subtracted from the other two models, and the results are illustrated in **Figure 1.**

Figure 1 illustrates that stock price has an obvious harmonic oscillator effect; thus, the difference in stock prices in the two models also has a harmonic oscillator effect. When the MS risk is included, the stock price is lower. This implies that the values of STS-MS-OnV and STS-MS-OnB are positive, and when the long-term equilibrium level is doubled, the effects of stock price differences are stronger than the doubling of the volatility; thus, the STS-MS-OnB line is above the STS-MS-OnV line.

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[1]. β and σ are constant in the STS model. The initial stock price is 50. $(\beta, \sigma, \kappa, \alpha, \omega, r)$ = (1.1, 0.25, 0.8, 0.005/3, 0.025, 0.05).

[2]. β is constant in the STS-MS-OnV model. The initial stock price is 50. $(\beta, \kappa, \alpha, \omega, r)$ = (1.1, 0.8, 0.005/3, 0.025, 0.05), $\sigma\big|_{e_1}$ = 0.25, and $\sigma\big|_{e_2}$ = 0.55.

[3]. σ is constant in the STS-MS-OnB model. The initial stock price is 50. $(\sigma, \kappa, \alpha, \omega, r)$ = (0.25, 0.8, 0.005/3, 0.025, 0.05), $\beta\big|_{e_1}$ = 1.1, and $\beta\big|_{e_2}$ = 2.2.

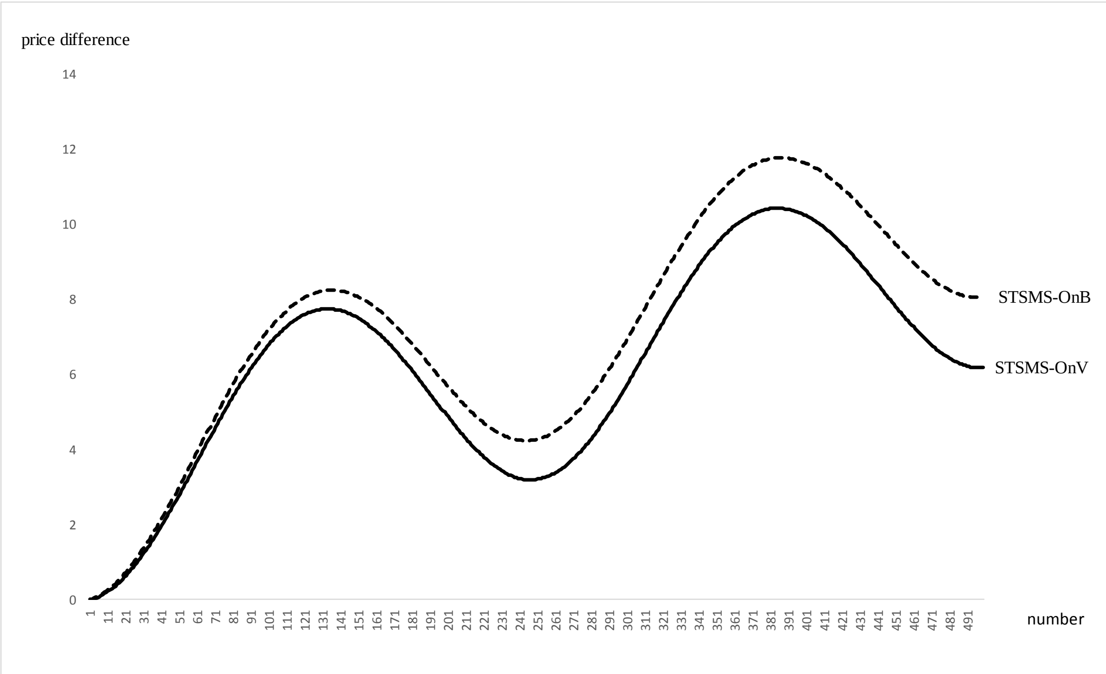

Figure 1. Simulated stock price difference path in the three models

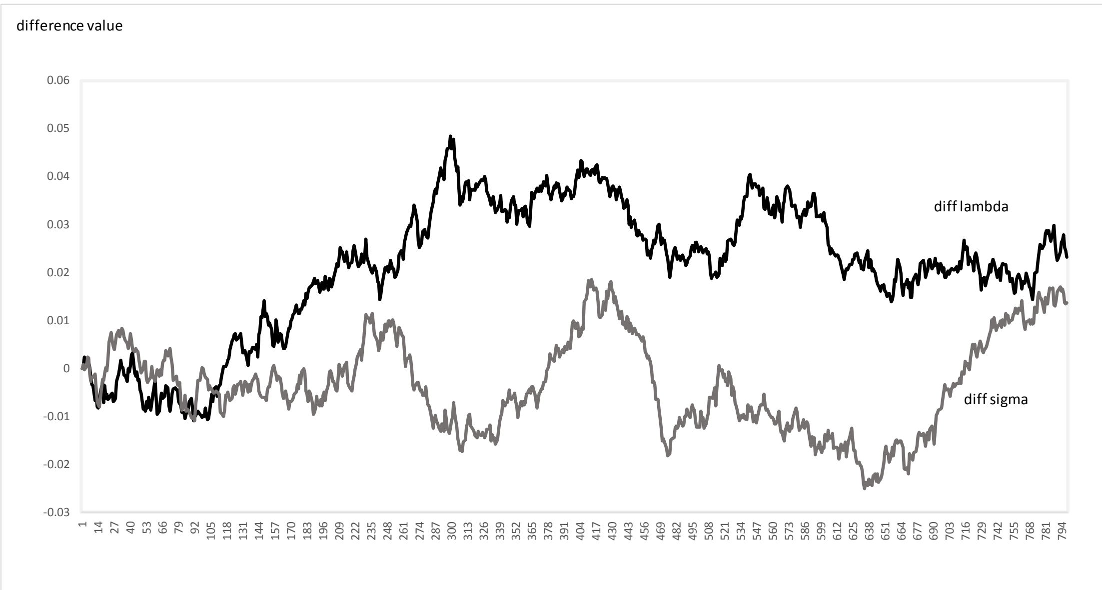

Figure 2. Difference in stock price in long- and short-term economies

In **Figure 2**, the “diff-lambda” curve shows the stock prices in a short-term economy minus the stock prices in a long-term economy. The curve is almost always greater than zero because the long-term equilibrium level is positive. Comparing “diff-sigma” and “diff-lambda” reveals that the uncertainty property caused by the Markov-switch risk is different from the uncertainty property caused by the volatility risk. This thus motivates us to expand the STS model into the STS-MS model.

5.2 Simulation of European call option price using the STS-MS model

In this section, we focus on the effect of a change in a business cycle on call option prices. In **Figure 3**, “diff_modelV” and “diff_modelB” respectively represent the call price difference between different economies in the STS-MS-OnV and STS-MS-OnB models; “diff_modeltwo” represents the call price difference between different economies in the STS-MS model.

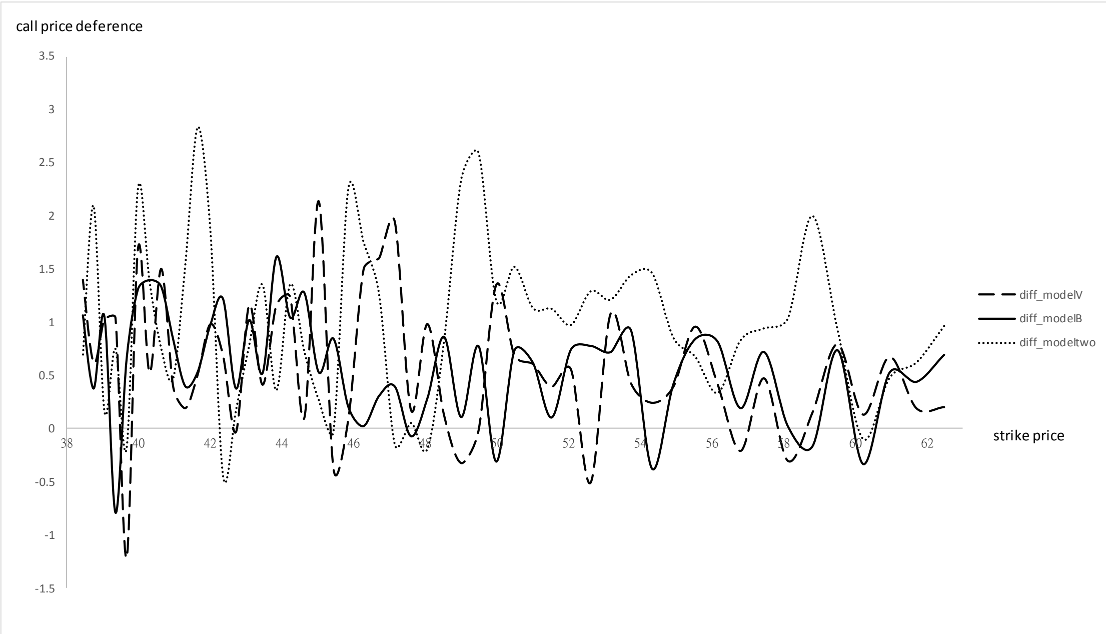

Figure 3. Difference in call price in long- and short-term economies

In **Figure 3**, the three lines are mostly above zero, indicating that the call price in the long-term economy is more frequently greater than that in the short-term economy. In addition, the call prices of the STS-MS model exhibit huge amplitudes. Therefore, the differences between differing economies are more easily expressed using the STS-MS model. This is in agreement with our expectations.

6. Conclusion

Under an STS-MS model that includes stochastic interest rates and stochastic assets, we derive the valuation of a European call option. To find the analytic formulas of the call option, we restrict the instantaneous forward rate process as a one-factor HJM model. Furthermore, the numerical analysis indicates that stock price characteristics in the STS-MS model and call price in the STS-MS model are well matched for different economies. Our results suggest several promising directions for future research. Some exotic options can be solved using our approach. The interplay between option and business cycle can be further analyzed. Based on the STS-MS model, an empirical study can conducted to examine related call options and provide valuable tools for option management under changing economic conditions.

Appendix A (proof of Proposition 1)

Proof: $$C_{Tra}(t;X^S)=e^{-r(t)\tau}E^Q\left[(S(T)-K)^+\left|F_t\right.\right]=e^{-r(t)\tau}\pi[K,\chi_S^Q]$$

$$\pi[K,\chi_S^Q]=E^Q\left[(S(T)-K)^+\left|F_t\right.\right]$$

$$=S(t)E^Q\left[(e^{\ln\frac{S(T)}{S(t)}}-e^{\ln\frac{K}{S(t)}})^+\left|F_t\right.\right]$$

$$=S(t)E^Q\left[(e^{y(T)}-e^{\varepsilon})^+\left|F_t\right.\right]. \quad (A1)$$

According to Proposition 1 in Moreno and Platania (2011), we have

$$y(T)=y(t)e^{-\kappa(T-t)}+\alpha\frac{\kappa^2}{\kappa^2-\omega^2}H(t,T)+(\beta(t)-\frac{\lambda(t)\sigma(t)}{\kappa})\left[1-e^{-\kappa(T-t)}\right]+\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_r^Q(u)$$

$$=\Theta(t,T)+\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_r^Q(u). \quad (A2)$$

Substituting Eq. (A2) into Eq. (A1) yields

$$\pi[K,\chi_S^Q]=S(t)E^Q\left[(e^{\left[\Theta(t,T)+\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u)\right]}-e^{\varepsilon})^+\left|F_t\right.\right]$$

$$=S(t)e^{\varepsilon}E^Q\left[(e^{x(T)-h}-1)^+\left|F_t\right.\right]$$

where $h=\varepsilon-\Theta(t,T)$ and $x(T)=\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u)$.

For a result similar to the following proof, see Raible (2000, pp. 64–66):

$$C_{Tra}(t;X^S)=e^{-r(t)\tau}\pi[K,\chi_S^Q]=S(t)e^{-r(t)\tau+\varepsilon}\frac{e^{hM}}{2\pi}\int_{-\infty}^{\infty}e^{ihu}\frac{\chi_S^Q(iM-u)}{(M+iu)(M+iu+1)}du,$$

where $\chi_S^Q(u)=E\left[\exp(iu\cdot\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u))\right]$. (A3)

Appendix B (proof of Proposition 2)

Proof: The result in Section 2.4 implies that

$$C_{Tra}(t;X^S)=P(t,T)E^{Q_T}\left[(S(T)-K)^+\left|\Im^S(t)\vee\Im^r(t)\right.\right]=P(t,T)\pi[K,\chi_S^{Q_T}]$$ and

$$\pi[K,\chi_S^{Q_T}]=E^{Q_T}\left[(S(T)-K)^+\left|F_t\right.\right]$$

$$=S(t)E^{Q_T}\left[(e^{y(T)}-e^{\varepsilon})^+\left|F_t\right.\right]$$

$$=S(t)e^{\varepsilon}E^{Q_T}\left[(e^{x(T)-h}-1)^+\left|F_t\right.\right]$$

$$=S(t)e^{\varepsilon}\frac{e^{hM}}{2\pi}\int_{-\infty}^{\infty}e^{ihu}\frac{\chi_S^{Q_T}(iM-u)}{(M+iu)(M+iu+1)}du.$$

In addition,

$$\chi_S^{Q_T}(u)=E^Q\left[\frac{dQ_T}{dQ}\exp(iu\cdot\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u))\right]$$

$$=E^Q\left[\frac{P(T,T)/B(T)}{P(0,T)/B(0)}\exp(iu\cdot\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u))\right]$$

$$=\exp\left(\int_0^T\left[-\frac{1}{2}\xi_2(s,T)^2\right]ds\right)E^Q\left[\exp(\int_0^T-\xi_2(s,T)dW_f^Q(s))\exp(z\sigma(t)\int_t^T e^{-\kappa(T-u)}dw_S{}^Q(u))\right]$$

$$=\exp\left(\int_0^T\left[-\frac{1}{2}\xi_2(s,T)^2\right]ds\right)\cdot\Psi(t).$$

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